B4.2 Functional Analysis II - Sheet 3 of 4

Reread Chapter 2, read Chapter 3 and prove the few statements whose proofs were left out as an exercise. (Not to be handed in.)

Do:

Q1. Let X and Y be real Hilbert spaces and $T \in \mathscr{B}(X, Y)$ be surjective. Show that there exists a unique bounded linear operator $R \in \mathscr{B}(Y, X)$ such that $TR = I_Y$ and $\|Ry\| \leq \|x\|$ for all $x \in X$ and $y \in Y$ satisfying $Tx = y$.

[*Hint: Follow the strategy of the proof of Theorem 2.3.4.*]

Q2. Let X be a Hilbert space and $T \in \mathcal{B}(X)$. Show that the graph $\Gamma(T)$ of T is a closed subspace of $X \times X$ and that

$$\Gamma(T)^{\perp} = \{(-T^{*}x, x) : \; x \in X\}.$$

By considering the corresponding orthogonal decomposition of $(x, 0)$, prove that $I + T^{*}T$ maps X onto X. Here the space $X \times X$ is endowed with the product inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{X \times X} = \langle x_1, y_1 \rangle_X + \langle x_2, y_2 \rangle_X.$$

Q3. Let X and Y be real Banach spaces and $T \in \mathscr{B}(X, Y)$. Assume that $Z = TX$ is a finite-codimensional subspace of Y and let $\{y_1 + Z, \ldots, y_m + Z\}$ be a basis for Y/Z. Define $\hat{T} : X \oplus \mathbb{R}^m \to Y$ by

$$\hat{T}(x, (v_1, \ldots, v_m)) = T(x) + \sum_{j=1}^{m} v_j \, y_j.$$

Show that $\hat{T}$ is a surjective bounded linear operator. Hence, by applying the open mapping theorem, deduce that Z is closed.

Q4. Let X be a Banach space.

(a) Show that if a sequence (x_n) in X converges weakly, then its weak limit is unique.

(b) Suppose that $x_n \rightharpoonup x$ in X and $\ell_n \to \ell$ in X^*. Show that $\ell_n(x_n) \to \ell(x)$.

(c) Suppose in addition that X is a Hilbert space. Show that if $x_n \rightharpoonup x$ in X and if $\|x_n\| \to \|x\|$, then $x_n \to x$.

(d) Prove (c) when the assumption that X is a Hilbert space is replaced by the assumption that X is uniformly convex: for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and if $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2(1 - \delta)$.

[*Hint: Consider the sequence* $\frac{1}{2}(x_n + x)$ *and use Theorem 3.2.2.*]

Q5. All sequence spaces in this question are real.

(a) Let $1 < p < \infty$. Show that a sequence $(x_n) \subset \ell^p$ converges weakly to x if and only if it is bounded and $x_n(j) \to x(j)$ for every j.

[*Hint: Use weak sequential compactness* or *the inequality*

$$\Big|\sum_j \alpha(j)\beta(j)\Big| \leq \Big\{\sum_j |\alpha(j)|^p\Big\}^{1/p}\Big\{\sum_j |\beta(j)|^q\Big\}^{1/q}. \qquad]$$

(b*) Show that a sequence in ℓ^1 is weakly convergent if and only if it is strongly convergent.

[*Hint: For given* $\varepsilon > 0$, *construct inductively increasing sequences* n_k *and* m_k *such that* $\sum_{j \leq m_{k-1}} |x_{n_k}(j)| < \varepsilon/8$ *and* $\sum_{j > m_k} |x_{n_k}(j)| < \varepsilon/8$. *Then test the weak convergence against* $b \in \ell^\infty$ *given by* $b(j) = sign(x_{n_k}(j))$ *for* $m_{k-1} < j \leq m_k$.]

(c) Let $1 \leq p < \infty$ and let $e_n \in \ell^p$ denote the sequence $(\delta_{nj})_{j=1}^\infty$ where δ is the Kronecker delta. Does (e_n) converge weakly or strongly in ℓ^p? If it converges (weakly or strongly), identify its limit.

Q6. A sequence (ℓ_n) in the dual space X^* of a Banach space X is said to be weak* convergent to $\ell \in X^*$ if

$$\ell_n(x) \to \ell(x) \text{ for all } x \in X.$$

(a) Show that weak* convergent sequences are bounded.

(b) Show that if X is separable, then the unit ball of X^* is weak* sequentially compact, i.e. every sequence (ℓ_n) in X^* with $\|\ell_n\|_* \leq 1$ has a weak* convergent subsequence.

[*Hint: Let* (x_n) *be a dense subset of* X. *Mimic the proof of the weak sequential compactness of the unit ball to construct a subsequence* (ℓ_{n_k}) *such that* $\ell_{n_k}(x_m)$ *is convergent for every* m.]