

## B4.2 Functional Analysis II - Sheet 4 of 4

Read Chapter 4 & 5 and prove the few statements whose proofs were left out as an exercise. (Not to be handed in.)

Do:

**Q0.** (Not to be handed in.) This problem recalls a result in Part A Integration.

Let  $f \in L^1_{loc}(\mathbb{R})$ ,  $a \in \mathbb{R}$  and define

$$F(x) = \int_a^x f(t) dt.$$

- (i) Show that  $F$  is continuous.
- (ii) Show that if  $\varphi$  is a smooth function, then integration by parts hold:

$$\int_b^c F(x) \varphi'(x) dx = [F\varphi]_b^c - \int_b^c f(x) \varphi(x) dx.$$

[Hint: First prove the statement for the case  $b = a$  by expressing the left hand side as a repeated integral and then appealing to Fubini's theorem.]

- (iii) Show that if  $F$  is constant, then  $f = 0$  a.e.

**Q1.** (a) Use Theorem 4.6.1 to prove the localisation property of Fourier series: if two (continuous)  $2\pi$ -periodic functions  $f$  and  $g$  are equal in an open interval containing 0, then their Fourier series either both converge at 0 or both diverge at 0.

- (b) In the lecture, we prove that there is a continuous function whose Fourier series diverges at 0. Use (a) to construct a continuous function whose Fourier series diverges at 0 and  $\pi/2$ .

**Q2.** Consider the system  $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}}$  as a subset of  $X = L^1(-\pi, \pi)$ .

- (a) Show that  $\|e_n\| = \sqrt{2\pi}$  for all  $n$  and  $\|e_n - e_m\| = \frac{8}{\sqrt{2\pi}}$  for all  $n \neq m$ .
- (b\*) Show that  $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}}$  is a basis of  $L^1(-\pi, \pi)$ , i.e. the closed linear span of  $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}}$  is  $L^1(-\pi, \pi)$ .

- Q3.** Let  $X$  be the closed subspace of  $C[-\pi, \pi]$  consisting of all continuous (on  $[-\pi, \pi]$ ) functions  $f$  such that  $f(-\pi) = f(\pi)$ . For  $n \in \mathbb{Z}$ , define  $e_n \in X$  by  $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$  and let

$$\widehat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

for  $f \in X$ . Let  $\{\alpha_n\}_{n \in \mathbb{Z}}$  be a sequence in  $\mathbb{C}$ , and assume that for each  $f \in X$  there exists a unique element  $g \in X$  such that  $\widehat{g}(n) = \alpha_n \widehat{f}(n)$  for all  $n \in \mathbb{Z}$ . Let  $Tf = g$ .

- (a) Show that  $T$  is linear and has closed graph. Deduce that  $T \in \mathcal{B}(X)$ .
  - (b) Show that  $Te_n = \alpha_n e_n$  for all  $n \in \mathbb{Z}$  and that the sequence  $\{\alpha_n\}_{n \in \mathbb{Z}}$  is bounded.
  - (c) Show that there exists a bounded linear functional  $\varphi$  on  $X$  such that  $\varphi(e_n) = \alpha_n$  for all  $n \in \mathbb{Z}$ .
- Q4.** Consider the right shift operator on sequences  $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ . Show that as an operator on  $\ell^2$ ,  $R$  satisfies  $\sigma_p(R) = \emptyset$ ,  $\sigma_r(R) = \{\lambda : |\lambda| < 1\}$  and  $\sigma_c(R) = \{\lambda : |\lambda| = 1\}$ .  
[To put thing in perspective, compare Question 7 of Sheet 4 of B4.1 from MT: If we consider  $T$  as an operator on  $\ell^\infty$ , then  $\sigma_p(R) = \emptyset$ ,  $\sigma_r(R) = \{\lambda : |\lambda| \leq 1\}$  and  $\sigma_c(R) = \emptyset$ .]
- Q5.** Let  $X$  be a complex Hilbert space and  $A \in \mathcal{B}(X)$  be normal (i.e.  $A^*A = AA^*$ ).

- (a) Show that

$$\text{rad}(\sigma(A)) = \|A\|.$$

Deduce that if  $P$  is a polynomial, then

$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|.$$

- (b) Let  $P$  be a Laurent polynomial, i.e.  $P(z) = \sum_k a_k z^k$  where the summation range is finite but may contains positive as well as negative powers. Show that if  $A$  is unitary, then

$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|.$$

**Q6.** Let  $X$  be a complex Hilbert space and  $S$  and  $T$  be two self-adjoint bounded linear operators on  $X$ .

- (a) Let  $\lambda \notin \sigma(T)$ . Use the fact that  $\sigma((T - \lambda I)^{-1}) = (\sigma(T) - \lambda)^{-1}$  (a form of spectral mapping theorem) and Gelfand's formula to show that

$$\|(T - \lambda I)^{-1}\| = \frac{1}{\text{dist}(\lambda, \sigma(T))}.$$

Deduce that  $I + (T - \lambda I)^{-1}(S - T)$  is invertible if

$$\|S - T\| < \text{dist}(\lambda, \sigma(T)).$$

Hence, show under this latter assumption that  $\lambda \notin \sigma(S)$ .

- (b) Use (a) to show that

$$\|S - T\| \geq \text{dist}_H(\sigma(S), \sigma(T))$$

where the Hausdorff distance  $\text{dist}_H(A, B)$  between two closed subsets  $A$  and  $B$  of  $\mathbb{C}$  is defined by

$$\text{dist}_H(A, B) = \max(\sup_{a \in A} \min_{b \in B} |a - b|, \sup_{b \in B} \min_{a \in A} |a - b|).$$