Algebraic Number Theory: Problem Sheet 1. 2019.

Topics covered: algebraic numbers, algebraic integers, norms, traces, complex embeddings, discriminants, Stickelberger's theorem, integral bases.

1. (a) Show that each of the following numbers is algebraic

$$1/2, \sqrt{-5}, \sqrt{17}+\sqrt{19}, e^{2\pi i/7}.$$

   (b) Assuming that the polynomials you have found are irreducible, what are the (absolute) conjugates of these numbers, and

   (c) Calculate their (absolute) traces and norms

2. (a) Let $K = \mathbb{Q}(\theta)$ where $\theta^2 = d$, $d \in \mathbb{Z}$ not a square. Describe the embeddings σ_1, σ_2 of K into $\mathbb{C}$. Are the fields $\sigma_1(K)$, $\sigma_2(K)$ different?

   (b) Let $K = \mathbb{Q}(\phi)$ where $\phi^3 = d$, $d \in \mathbb{Z}$ not a cube. Describe the embeddings $\sigma_1, \sigma_2, \sigma_3$ of K into $\mathbb{C}$. Are the fields $\sigma_1(K), \sigma_2(K), \sigma_3(K)$ different?

3. Let $K = \mathbb{Q}(\alpha)$, $\alpha^3 = m$, m not a cube. Evaluate $\Delta(1, \alpha, \alpha^2)^2$ by the formula $\Delta = \det(\sigma_i w_j)$. Write down the traces of $1, \alpha, \ldots, \alpha^4$ and hence evaluate $\Delta(1, \alpha, \alpha^2)^2$ by the formula involving traces.

4. Suppose that β is a root of $X^3 + pX + q = 0$, where $X^3 + pX + q$ is an irreducible polynomial in $\mathbb{Z}[X]$. Verify that $1, \beta, \beta^2, \beta^3$ have traces $3, 0, -2p, -3q$, respectively, and compute $\mathrm{Tr}(\beta^4)$. Deduce that $\Delta(1, \beta, \beta^2)^2 = -4p^3 - 27q^2$.

5. Suppose that α is a root of a monic irreducible polynomial $f(X) \in \mathbb{Z}[X]$. Prove that if $\deg(f) = n$ and $K = \mathbb{Q}(\alpha)$ then

$$\Delta^2(1, \alpha, \ldots, \alpha^{n-1}) = (-1)^{n(n-1)/2}\mathrm{Norm}_{K/\mathbb{Q}}(f'(\alpha)).$$

6. Suppose that $[K : \mathbb{Q}] = n$, and that there are r real embeddings and s pairs of complex embeddings of K into $\mathbb{C}$, where $r + 2s = n$. Show that if $w = \{w_1, \ldots, w_n\}$ is an integral basis for $\mathcal{O}_K$ then the sign of $\Delta(w)^2$ is $(-1)^s$.

7. [Stickelberger's Theorem] With the notation of the preceding question, let M be a splitting field containing K. Write Ω for the matrix $(\sigma_i(w_j))$, and write P for the sum of the terms in the expansion of $\det(\Omega)$ that occur with positive sign, and N for the sum of the terms which occur with negative sign; so $\Delta(w) = P - N$ and $P + N$ is the "permanent". Show that $P + N$ and PN are both invariant by $\mathrm{Gal}(M/\mathbb{Q})$, so are both rational integers. Deduce that $\Delta(K)^2 \equiv 0, 1 \bmod 4$.

Further Practice: Exercises in Chapter 2 of Stewart and Tall.