Professor Joyce **B3.3 Algebraic Curves** **Hilary Term 2019**

Problem Sheet 2

1. Show that the curve $y^2z = x^3 + xz^2$ in $\mathbb{CP}^2$ is nonsingular.

Now consider this curve over the finite field $\mathbb{Z}_p$, where p is a prime. That is, we consider the curve in $P\big((\mathbb{Z}_p)^3\big)$ with equation $y^2z = x^3 + xz^2$. For which p is this nonsingular?

2. Let A and B be two symmetric 3×3 complex matrices and suppose the equation $\det(xA - B) = 0$ has three distinct solutions λ, μ, ν.

(i) Show that there is an invertible matrix P such that $P^TAP = I$ and $P^TBP = \mathrm{diag}(\lambda, \mu, \nu)$.

(ii) Deduce that, after a projective transformation, the equations of the conics defined by A and B can be put in the form

$$x^2 + y^2 + z^2 = 0, \qquad \lambda x^2 + \mu y^2 + \nu z^2 = 0.$$

(iii) Show that these two conics intersect in four distinct points.

3. This question deals with how to define tangent lines at singular points. For simplicity we work in $\mathbb{C}^2$ rather than $\mathbb{CP}^2$.

Let C be a curve in $\mathbb{C}^2$ defined by $Q(x, y) = 0$ for $Q(x, y)$ a complex polynomial without repeated factors. Define the *multiplicity* m of C at a point $(a, b) \in C$ to be the smallest positive integer m such that some m^{th} partial derivative of Q at (a, b) is nonzero (so (a, b) is a singularity of C if and only if $m > 1$).

Consider the polynomial

$$\sum_{i+j=m} \frac{\partial^m Q}{\partial x^i \partial y^j}(a, b)\frac{(x-a)^i(y-b)^j}{i!j!}.$$

(i) Show this factorizes as a product of m linear polynomials of the form

$$\alpha(x - a) + \beta(y - b).$$

The lines defined by the vanishing of these linear polynomials are called the *m tangent lines* to C at (a, b).

(ii) Show that if $m = 1$ this definition agrees with the definition given in lectures for the tangent line at a nonsingular point.

(iii) For the nodal cubic $y^2 = x^3 + x^2$ and the cuspidal cubic $y^2 = x^3$, find all the singular points in $\mathbb{C}^2$, and for each singular point, find the multiplicities and tangent lines.

4. Find all the singular points, and for each singular point, find the multiplicities and tangent lines (in the sense of Problem 3), of the following affine curves:

(i) the cuspidal cubic $y^2 = x^3$;

(ii) the nodal cubic $y^2 = x^3 + x^2$;

(iii) the quartic curve $x^4 + y^4 = 2xy^2$;

(iv) the sextic curve $(x^2 + y^2)^3 = 4x^2y^2$.

Sketch real pictures for each of these curves and draw all the real tangent lines at the singular points.

DDJ