Professor Joyce **B3.3 Algebraic Curves** **Hilary Term 2019**

Problem Sheet 4

1. Let C be a Riemann surface, and $f, g : C \to \mathbb{CP}^1 = \mathbb{C} \amalg \{\infty\}$ be meromorphic functions such that $f, g \not\equiv \infty$, and $\lambda \in \mathbb{C}$. Show that $fg, f+g$ and λf are also meromorphic functions on C (or at least, can be extended to meromorphic functions). Deduce that the meromorphic functions f on C with $f \not\equiv \infty$ form a *commutative algebra* over $\mathbb{C}$.

Note: the issue here is that algebraic operations of addition, multiplication and scalar multiplication are defined on $\mathbb{C}$, but not on $\mathbb{C} \amalg \{\infty\}$. So, for example, $f(p)g(p)$ is not well-defined at a point $p \in C$ with $f(p) = \infty$ and $g(p) = 0$. You should prove that $fg, f+g$ and λf are well-defined except at isolated points in C, and they extend uniquely over these points to give meromorphic functions.

2. Let $\wp(z)$ be the Weierstrass $\wp$-function. Consider the meromorphic function $\wp'(z)$ on $X = \mathbb{C}/\Lambda$. By considering $\wp'$ as a map to $\mathbb{CP}^1$, determine its degree and the number and indices of its ramification points.

Is there a meromorphic function f on X such that $f'(z) = \wp(z)$?

[*Hint: What would its poles look like?*]

3. The $\wp$-function satisfies $\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$. Supposing that none of e_1, e_2, e_3 is zero, show that the equation $\wp(z) = 0$ has two distinct solutions $z = \pm a$. For any two distinct points $b, c \in \mathbb{C}/\Gamma$ write down a meromorphic function whose only poles are simple poles at b and c.

4. Let C be a nonsingular cubic curve and let $p, q \in C$ be distinct points. Show using the Riemann–Roch Theorem that $\ell(p+q) = 2$. Deduce that there exists a meromorphic function on C whose only poles are simple poles at p and q.

5. Let C be a nonsingular quartic curve in $\mathbb{CP}^2$.

- **(a)** Let H be a hyperplane divisor on C. Show that $\ell(H) \geq 3$.
- **(b)** Let D be a divisor on C with $\deg D = 4$. Prove using the Riemann–Roch Theorem that $\ell(D) = 3$ if D is a canonical divisor, and $\ell(D) = 2$ otherwise.
- **(c)** Deduce that every hyperplane divisor on C is a canonical divisor.

6. (Optional). Let C be a nonsingular quartic curve in $\mathbb{CP}^2$.

(i) Let $HD(C)$ be the vector space of *holomorphic differentials* on C. Show using question 5 that there is an isomorphism $HD(C) \cong \mathbb{C}^3 = \langle x, y, z \rangle$, such that if $0 \neq \omega$ is a holomorphic differential corresponding to $ax + by + cz$ then the canonical divisor (ω) is the hyperplane divisor corresponding to the line $ax + by + cz = 0$.

(ii) Let $p, q \in C$ be distinct points. Show that the vector subspace of $\omega \in HD(C)$ vanishing at p, q has dimension 1. Deduce that $\ell(\kappa - p - q) = 1$ for a canonical divisor κ.

(iii) Show that $\ell(p + q) = 1$ for all distinct $p, q \in C$.

7. (Optional). Consider the affine nodal cubic C_{aff} in $\mathbb{C}^2$ with equation

$$y^2 = x^3 + x^2.$$

Show that the formula

$$t \mapsto (t^2 - 1, t - t^3)$$

describes a map from $\mathbb{C}$ onto C_{aff}. Describe the fibres of this map (i.e. the preimages of points in C_{aff}).

What can you deduce about the topology of the projective nodal cubic $y^2 z = x^3 + x^2 z$ in $\mathbb{CP}^2$?

8. (Optional). This problem shows how to associate a ring to a divisor on a curve, and studies this ring in a relatively simple but typical example.

(a) Let C be a projective a nonsingular curve in $\mathbb{CP}^2$, and D a divisor on C. Show that the vector space

$$R(D) = \bigoplus_{n \geq 0} \mathcal{L}(nD)$$

can naturally be given the structure of a (graded, if you know what that is) ring.

(b) Let C be a nonsingular cubic curve, and $D = p$ for a point $p \in C$. Compute the dimension $\ell(nD)$ for each $n \geq 0$.

(c) Using your computations, study the structure of the ring $R(D)$ in this example in the following terms.

(i) In degree $n = 0$, we have the constants $\mathbb{C}$.

(ii) In degree $n = 1$, we have a one-dimensional vector space $\mathcal{L}(D)$, generated by an element we call $x \in R(D)$. [This is a confusing point. As a meromorphic function, x is the constant function 1. However, as an element of $R(D)$, it is different from the identity element, hence we need to give it a different name!]

(iii) In degree $n = 2$, we have a two-dimensional vector space $\mathcal{L}(2D)$, which has a basis element x^2 [still represented by the constant function!] and a new basis element that we call y.

(iv) In degree $n = 3$, we have a three-dimensional vector space $\mathcal{L}(3D)$, of which we know two elements x^3 and xy. Assuming these are linearly independent, we need one more basis element z.

(v) Show that, making the appropriate linear independence assumptions, in degrees $n = 4$ and 5 there is no need for any further generators, the known elements of the ring exactly span the vector spaces $\mathcal{L}(nD)$.

(vi) Show however that in degree $n = 6$, there are too many known elements in $\mathcal{L}(6D)$, leading to a linear dependence relation between expressions in the quantities x, y, z. Show that, after a change of generators, we can assume this relation to be of the form

$$z^2 = y^3 + ax^4y + bx^6. \tag{1}$$

(vii) Show that, assuming that the elements x, y, z generate the ring $R(D)$, relation (1) is the only relation there exists in $R(D)$: the appropriate combination of x, y, z's gives a vector space which is exactly the right dimension for every n.

(d) If you have enjoyed the story so far, repeat the analysis for the divisor $D = 2p$ on the nonsingular cubic curve C.

DDJ