Combinatorics C8.3 **Problem 0 solutions** **EL MT2018**

The aim of this sheet is to highlight some useful results and arguments in combinatorics. The problems make use of a small amount of graph theory, but everything required can be found in the subsection on Hall's theorem in Section 7 of the Part B Graph Theory notes. There will not be tutorials on this problem sheet, but complete solutions are available on the course webpage. All graphs below are assumed to be finite.

Remark: The combinatorics course is essentially self-contained, so do not feel put off if there is notation below which you haven't encountered before.

1. Let G be a bipartite graph with bipartition (A, B). Suppose also that every vertex in G has the same degree $d > 0$.

   (a) Show that $|A| = |B|$.

   (b) Look up Hall's theorem. Use this result to prove that G contains a complete matching.

   (c) Show that the edge set of G can be partitioned into d edge disjoint complete matchings.

Solution: For part (a), note that since G is bipartite we can count the edges of G by summing the degrees in either A or B. This gives

$$d|A| = \sum_{a\in A} d_G(a) = e(G) = \sum_{b\in B} d_G(b) = d|B|.$$

As $d > 0$ this implies $|A| = |B|$.

(b) Given $S \subset A$ let $\Gamma(S) := \{b \in B : ab \in E(G) \text{ for some } a \in S\} \subset B$. By Hall's theorem in order to prove G has a complete matching from A to B it is enough to show that $|\Gamma(S)| \geq |S|$ for all $S \subset A$. To see this, we will estimate edges between S and $\Gamma(S)$ in two ways; this technique is often referred to as 'double counting'. Note that

$$d|S| = \left|\{(a, b) \in S \times \Gamma(S) : ab \in E(G)\}\right| \leq d|\Gamma(S)|.$$

The equality holds as each $a \in S$ has all $d_G(a) = d$ neighbours in $\Gamma(S)$ and the inequality holds since each $b \in \Gamma(S)$ has at most $d_G(b) = d$ neighbours in S. Dividing by d we see that the conditions of Hall's theorem hold for G.

(c) By induction on d. If $d = 1$ then the edges of G form a complete matching. We will prove the result for $d \geq 2$ assuming by induction that it holds for smaller degree. From (b) there is a complete matching $\mathcal{M}$ in G. Let G' denote the graph obtained from G by deleting the edges of $\mathcal{M}$. All vertices in G' have degree $d-1$ and so the edges of G' can be partitioned into $d-1$ complete matching $\mathcal{M}_1, \ldots, \mathcal{M}_{d-1}$. Combined with $\mathcal{M}$ this gives the required partition.

2. Let $[n]^{(i)} := \big\{A \subset \{1, \ldots, n\} : |A| = i\big\}$ and suppose that $i < n/2$. Prove that for each $A \in [n]^{(i)}$ we can choose a set $B_A \in [n]^{(i+1)}$ so that $A \subset B_A$ for all A and such that the sets $\{B_A\}_A$ are all distinct.

Solution: Consider the bipartite graph G with bipartition $([n]^{(i)}, [n]^{(i+1)})$ in which $AB \in E(G)$ if $A \subset B$. The question is equivalent to proving that G contains a complete matching from $[n]^{(i)}$ to $[n]^{(i+1)}$. By Hall's theorem such a matching exists provided $|\Gamma(S)| \geq |S|$ for every $S \subset [n]^{(i)}$. We will prove this again by double counting the edges of G between S and $\Gamma(S)$.

First note that each $A \in [n]^{(i)}$ has exactly $n - i$ neighbours in G, one for each set $A \cup \{a'\}$ with $a' \in [n] \setminus A$. On the other hand, every vertex $B \in \Gamma(S)$ has at most $|B| = i + 1$ neighbours in $[n]^{(i)}$, one for each $B - \{b\}$. It follows that

$$|S|(n-i) = \sum_{A \in S} d_G(A) = e_G(S, \Gamma(S)) \leq \sum_{B \in \Gamma(S)} |B| = (i+1)||\Gamma(S)|.$$

Since $i < n/2$ this gives $|\Gamma(S)| \geq \left(\frac{i+1}{n-i}\right)|\Gamma(S)| \geq |S|$. Thus the conditions of Hall's theorem are satisfied and so the required matching exists.

3. Let G be a bipartite graph with bipartition (A, B) which contains a complete matching from A to B. Prove that there is $a \in A$ such that every edge $ab \in E(G)$ lies in a complete matching from A to B.

   Hint: Read the 'direct proof' of Hall's theorem in the Part B notes.

Solution: As indicated in the hint, this solution is based on the 'direct proof' of Hall's theorem in the Graph Theory Part B notes. While the main idea is small modification of this proof, for completeness I've included all details.

We will prove the statement by induction on $|A|$. When $|A| = 1$ it is immediate, so we will assume $|A| \geq 2$ and that the result holds for all smaller graphs. Since G has a complete matching from A to B, Hall's condition is satisfied for G (as it is necessary condition). Thus $|\Gamma(S)| \geq |S|$ for all $S \subset A$.

First suppose that

$$|\Gamma(S)| \geq |S| + 1, \tag{1}$$

for all $\emptyset \neq S \subsetneq A$. In this case we prove more: *every* edge $ab \in E(G)$ is contained in a complete matching. To see this, let G' denote the graph obtained from G by deleting the vertices a and b and all edges adjacent to these vertices. Then G' satisfies Hall's condition since for any $\emptyset \neq S \subset A' \backslash \{a\}$ we have $|\Gamma'(S)| \geq |\Gamma(S)| - |\{b\}| \geq (|S|+1) - 1 = |S|$, by (1). Thus G' contains a complete matching $\mathcal{M}'$ from $A \setminus \{a\}$ to $B \setminus \{b\}$. Combined with the edge ab, this gives a complete matching $\mathcal{M}$ from A to B containing the edge ab.

Suppose now instead that $|\Gamma(S)| = |S|$ for some $\emptyset \neq S \subsetneq A$. Let G_1 denote the induced bipartite subgraph of G with bipartition $(S, \Gamma(S))$. This graph satisfies Hall's conditions (since G does) and so by induction there is $a \in S$ so that every $ab \in E(G_1)$ appears in a complete matching $\mathcal{M}_1$ in G_1.

Claim: The vertex a also works for G.

To see this, take any $ab \in E(G)$. As $a \in S$, the neighbourhood of a is contained in $\Gamma(S)$ and so $ab \in E(G_1)$. Let $\mathcal{M}_1$ be a complete matching in G_1 containing ab. To complete the proof it is now enough to show that $\mathcal{M}_1$ can be extended to a complete matching $\mathcal{M}$ of G from A to B.

To see this, let G_2 be the induced subgraph of G which has bipartition $\big(A \setminus S, B \setminus \Gamma(S)\big)$. For any $S' \subset A \setminus S$ we have

$$|\Gamma_{G_2}(S')| \geq |\Gamma_G(S' \cup S)| - |\Gamma_G(S)| \geq |S' \cup S| - |S| = |S'|.$$

Thus Hall's condition holds for G_2. By induction G_2 contains a complete matching $\mathcal{M}_2$ from $A \setminus S$ to $B \setminus \Gamma(S)$. Setting $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$ then gives a complete matching from A to B which contains $\mathcal{M}_1$, as required.

4. Let $\mathcal{P}[n]$ denote the power set of $[n] := \{1, \ldots n\}$.

   (a) Prove that $|\mathcal{P}[n]| = 2^n$.

   (b) Select a set $A \in \mathcal{P}[n]$ uniformly at random. Let X denote the random variable given by $X(A) := |A|$. Prove that $\mathbb{E}(X) = n/2$ and $\mathbb{V}ar(X) = n/4$.

(c) Use Chebyshev's inequality and (b) to show that given $\epsilon > 0$ there is $C > 0$ such that $(1-\epsilon)2^n$ sets $A \subset \{1, \dots, n\}$ satisfy $\big||A| - \frac{n}{2}\big| \le C\sqrt{n}$.

Solution. (a) The map which sends the vector $(x_1, \dots, x_n) \in \{0,1\}^n$ to the set $\{i \in [n] : x_i = 1\} \in \mathcal{P}[n]$ is a bijection, and so $|\mathcal{P}[n]| = |\{0,1\}^n| = 2^n$.

To see (b), note that X can be written as a sum of indicator random variables $X = \sum_{i\in[n]} X_i$, where $X_i(A) = 1$ if $i \in A$ and $X_i(A) = 0$ if $i \notin A$. We have $\mathbb{E}(X_i) = \mathbb{P}(i \in A) = \frac{1}{2}$ for each $i \in [n]$ and so linearity of expectation gives

$$\mathbb{E}(X) = \mathbb{E}(\sum_{i\in[n]} X_i) = \sum_{i\in[n]} \mathbb{E}(X_i) = \frac{n}{2}.$$

To see the variance calculation recall that

$$\begin{aligned}\mathbb{V}ar(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 &= \mathbb{E}\Big((\sum_{i\in[n]} X_i)(\sum_{j\in[n]} X_j)\Big) - \frac{n^2}{4} \\ &= \sum_{i\in[n]} \mathbb{E}\Big(X_i^2\Big) + \sum_{i,j\in[n]:i\neq j} \mathbb{E}\Big(X_iX_j\Big) - \frac{n^2}{4} \\ &= n \cdot \frac{1}{2} + n(n-1)\cdot\frac{1}{4} - \frac{n^2}{4} = \frac{n}{4}.\end{aligned}$$

Here we used $\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \frac{1}{2}$ and $\mathbb{E}(X_iX_j) = \mathbb{P}(i, j \in A) = \frac{1}{4}$ for $i \neq j$.

For (c) note that for any $t > 0$ by Chebyshev's inequality we have

$$\mathbb{P}\Big(\big||A| - \frac{n}{2}\big| \ge t\Big) = \mathbb{P}\Big(\big|X - \mathbb{E}(X)\big| \ge t\Big) \le \frac{\mathbb{V}ar(X)}{t^2} = \frac{n}{4t^2}.$$

If we set $t = Cn^{1/2}$ where $C = \epsilon^{-1/2}/2$ then gives $\mathbb{P}\big(\big||A| - \frac{n}{2}\big| \ge t\big) \le \epsilon$. Since the sets were selected uniformly at random, this is equivalent to the statement that at most $\epsilon 2^n$ sets $A \in \mathcal{P}[n]$ satisfy $\big||A| - \frac{n}{2}\big| \ge t$.