# Combinatorics C8.3 – Problem sheet 4 solutions

1. Show that, for some $c > 1$ and every $n \geq 3$, there is a family $\mathcal{F} \subset \mathcal{P}[n]$ of size at least $c^n$ such that every set in $\mathcal{F}$ has odd size, and the intersection of any two distinct sets from $\mathcal{F}$ has odd size.

   Set $m = \lceil n/2 \rceil - 1 \geq n/4$ as $n \geq 3$. We take
   $$\mathcal{F} := \Big\{ \big( \cup_{i \in S} \{2i-1, 2i\} \big) \cup \{2m+1\} : S \subset [m] \Big\}.$$
   Then $|\mathcal{F}| = 2^m \geq 2^{n/4}$, and all sets and their intersections are odd.

2. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}[n]$ be two set systems such that $|A \cap B|$ is even for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^n$. Can you describe the pairs $\mathcal{A}$, $\mathcal{B}$ for which we have equality? (Hint: Show that if $A, A' \in \mathcal{A}$ then we may assume $A \triangle A' \in \mathcal{A}$.)

   We may assume $\mathcal{A}$ and $\mathcal{B}$ are maximal subject to the condition that $|A \cap B|$ is even for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then for all $A, A' \in \mathcal{A}$ we have $A \triangle A' \in \mathcal{A}$, as for $B \in \mathcal{B}$ we have
   $$|(A \triangle A') \cap B| = |A \cap B| + |A' \cap B| - 2|(A \cap A') \triangle B| \equiv 0 \mod 2.$$
   Similarly $B \triangle B' \in \mathcal{B}$ for all $B, B' \in \mathcal{B}$.

   View the characteristic vectors $V = \{\chi_A\}_{A \in \mathcal{A}}$ and $W = \{\chi_B\}_{B \in \mathcal{B}}$ as vectors in $\mathbb{F}_2^n$. As $\chi_{A \triangle A'} = \chi_A + \chi_{A'}$ in $\mathbb{F}_2^n$, by the previous paragraph we see that $V$ is a linear subspace of $\mathbb{F}_2^n$, and similarly with $W$. Furthermore, letting $\langle x, y \rangle$ denote the usual inner product on $\mathbb{F}_2^n$ we have
   $$\langle \chi_A, \chi_B \rangle \equiv |A \cap B| \equiv 0 \mod 2$$
   for all $\chi_A \in V$ and $\chi_B \in W$. Thus $V$ and $W$ are orthogonal subspaces. As $\langle \cdot, \cdot \rangle$ is non-degenerate, it follows that $\dim(V) + \dim(W) \leq n$, and so $|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^{\dim(V)} \cdot 2^{\dim(W)} \leq 2^n$.

   From above equality holds if and only if the characteristic vectors $\{\chi_A\}_{A \in \mathcal{A}}$ form a linear subspace $V$ of $\mathbb{F}_2^n$ and $\{\chi_B\}_{B \in \mathcal{B}} = \{\mathbf{w} \in \mathbb{F}_2^n : \langle v, w \rangle = 0 \text{ for all } v \in V\} = V^{\perp}$.

3. Prove that for $n \geq n_0(k)$ every family $\mathcal{A} \subset \binom{[n]}{k}$ which does not contain three disjoint sets satisfies $|\mathcal{A}| \leq |\mathcal{B}(n)|$, where
   $$\mathcal{B}(n) := \big\{ A \in [n]^{(k)} : A \cap \{1, 2\} \neq \emptyset \big\}.$$

The following solution follows a 'degrees of freedom' argument, similar to Theorem 23 in the notes. Provided $n \geq n_0(k)$, we will prove that if $\mathcal{A}$ does not contain three disjoint sets and $|\mathcal{A}| \geq |\mathcal{B}(n)|$ then there are distinct $x, y \in [n]$ and sets $A_1, \ldots, A_{k+1}, B_1, \ldots, B_{k+1} \in \mathcal{A}$ such that:

(a) $A_i \cap A_j = \{x\}$ for all $i, j \in [k+1]$ distinct;

(b) $B_i \cap B_j = \{y\}$ for all $i, j \in [k+1]$ distinct;

(c) $A_i \cap B_j = \emptyset$ for all $i, j \in [k+1]$.

This will prove the theorem. Indeed, if $C \in [n]^{(k)}$ and $C \cap \{x, y\} = \emptyset$ then $C$ is disjoint from some set $A_i$ and some set $B_j$, and so $C \notin \mathcal{A}$. This implies $\mathcal{A} \subset \{A \in [n]^{(k)} : A \cap \{x, y\} \neq \emptyset\}$, and so $|\mathcal{A}| \leq |\mathcal{B}(n)|$.

To begin, note that $|\mathcal{A}| \geq |\mathcal{B}(n)| > \binom{n-1}{k-1}$ and so by the Erdős–Ko–Rado theorem $\mathcal{A}$ contains two disjoint sets $A_1$ and $B_1$ say.

Suppose now that we have found $A_1, \ldots, A_\ell$ and $B_1, \ldots, B_\ell$ for some $\ell \in [k]$ and wish to extend the sequence, while satisfying (a), (b) and (c) above. Every set $C \in \mathcal{A}$ must intersect $A_1 \cup B_1$ in at least one element. Furthermore, the number of sets in $[n]^{(k)}$ which intersect $X = \cup_{i \leq \ell}(A_i \cup B_i)$ in more than one element is at most

$$\sum_{i=2}^{k} \binom{|X|}{i} \binom{n - |X|}{k - i} \leq \sum_{i=2}^{k} \binom{2k^2}{i} n^{k-2} \leq 2^{2k^2} n^{k-2}.$$

Therefore at least $|\mathcal{A}| - 2^{2k^2} n^{k-2} > \binom{n-1}{k-1}$ sets in $\mathcal{A}$ intersect $X$ in *exactly* one element (here we have used that $n \geq n_0(k)$). Again by the Erdős–Ko–Rado theorem there are two disjoint set $C$ and $D$ which intersect $X$ in one element. Consider two cases:

- $\ell = 1$: Then $C$ meets exactly one of $A_1$ and $B_1$; we may assume it meets $A_1$. Set $A_2 = C$ and set $x \in [n]$ to equal the element in $A_1 \cap A_2$. The set $D$ also meets $X$ in exactly one element and this must lie in $B_1$; otherwise $A_2, B_1, D \in \mathcal{A}$ and are disjoint. Set $B_2 = D$ and let $y \in [n]$ be the element with $\{y\} = B_1 \cap B_2$.
- $\ell \geq 2$: In this case, $x$ and $y$ have already been specified, e.g. $\{x\} = A_1 \cap A_2$. Note that if $C \cap \{x, y\} = \emptyset$ then there is some $A_i$ and $B_j$ which is disjoint from $C$, using that $C$ intersects $X$ in exactly one element, that $\ell \geq 2$ and that (a) and (b) hold. Therefore $x \in C$ (say) and we set $A_{\ell+1} = C$. The same argument guarantees that $D \cap \{x, y\} \neq \emptyset$. As $C$ and $D$ are disjoint $y \in D$ and we set $B_{\ell+1} = D$.

4. Let $P$ be a set of $n$ points in the plane that do not all lie on a straight line. Prove that they determine at least $n$ lines. [Hint: For each point, consider the set of lines that passes through it.]

The following solution is based on Fisher's inequality. Let $L = \{\ell_1, \ldots, \ell_m\}$ denote the set of lines containing at least two points from $P$. We want to prove that $m \geq n$. To see this, to each point $p \in P$ associate a set $A_p = \{i \in [m] : p \in \ell_i\} \subset [m]$. As no line contains all points from $P$ we have $|A_p| \geq 2$ for all $p \in P$. As any two distinct points $p, p' \in P$ lie on exactly one line, this gives $|A_p \cap A_{p'}| = 1$. It follows that $\mathcal{A} = \{A_p : p \in P\} \subset \mathcal{P}[m]$ satisfies $|\mathcal{A}| = |P| = n$ and by Fisher's inequality $n = |\mathcal{A}| \leq m$.

5. Prove that a non-trivial decomposition of the edges of $K_n$ into edge-disjoint complete subgraphs requires at least $n$ subgraphs. Show how this bound can be achieved.

   We will again prove this using Fisher's inequality. Let $C_1, \ldots, C_m$ denote the collection of complete subgraphs decomposing $K_n$. Let $A_x := \{i \in [m] : x \in C_i\} \subset [m]$ for each $x \in V(K_n)$. Note that as $C_1, \ldots, C_m$ forms a *non-trivial* decomposition we must have $|A_x| \geq 2$ for all $x \in V(K_n)$. Set $\mathcal{A} = \{A_x : x \in V(K_n)\} \subset \mathcal{P}[m]$.

   Given distinct $x, x' \in V(K_n)$, we have $i \in A_x \cap A_{x'}$ if and only if the edge $xx' \in E(C_i)$. It follows from the hypothesis that $|A_x \cap A_{x'}| = 1$ for all distinct $x, x' \in V(K_n)$. In particular, as $|A_x| \geq 2$ for all $x \in V(K_n)$, the sets in $\mathcal{A}$ are distinct and $|\mathcal{A}| = n$. By Fisher's inequality it follows that $n = |\mathcal{A}| \leq m$, as required.

   To see that the bound can be achieved, pick any vertex $x_0 \in V(K_n)$. Let $C_1$ denote the clique containing the vertex set $V(K_n) \setminus \{x_0\}$ and let $C_2, \ldots, C_n$ denote the remaining cliques of size two (i.e. edges) containing to $x_0$.

6. A set $P$ in $\mathbb{R}^n$ is a *two-distance set* if there are real numbers $\alpha, \beta$ such that $||x-y||_2 \in \{\alpha, \beta\}$ for all distinct $x, y \in P$. Let $P = \{p_1, \ldots, p_k\}$ be a two-distance set.

   (a) For each $i \in [k]$, let $f_i$ be the polynomial in variables $x = (x_1, \ldots, x_n)$ defined by

   $$f_i(x) = (||x - p_i||_2^2 - \alpha^2)(||x - p_i||_2^2 - \beta^2).$$

   Show that the polynomials $f_i$ are linearly independent. [Hint: Consider $f_i(p_j)$.]

   (b) Deduce that $k \leq \binom{n}{2} + 3n + 2$. [Hint: Find a basis for the space spanned by the polynomials $f_i$.]

   (a) First note that we may assume $\alpha, \beta \neq 0$, as $\|x - y\|_2 \neq 0$ for all distinct $x, y \in \mathbb{R}^n$.

   Suppose that $\sum_{i \in [k]} \lambda_i f_i(x) = 0$, where $\lambda_i \in \mathbb{R}$ for all $i \in [k]$. Substituting $p_j$ into this expression we obtain

   $$\alpha^2 \beta^2 \lambda_j + 0 = \sum_{i \in [k]} \lambda_i f_i(p_j) = 0.$$

   Here we have used that $f_i(p_j) = \alpha^2 \beta^2$ if $i = j$ and equals 0 if $i \neq j$. It follows that $\lambda_j = 0$ (since $\alpha, \beta \neq 0$) for all $j \in [k]$. Thus $f_1, \ldots, f_k$ are linearly independent, as required.

   (b) For any $q \in \mathbb{R}^n$ we can write $||x - q||_2^2 - \alpha^2 = ||x||_2^2 - \sum_{j \in [n]} 2q_i x_i + (||q||_2^2 - \alpha^2)$, and so $||x - q||_2^2 - \alpha^2$ can be written as a linear combination of the functions

   $$||x||_2^2, \quad \{x_i\}_{i \in [n]} \quad \text{and} \quad 1.$$

It follows that each function $f_i$ can be written as a linear combination of the functions

$$||x||_2^4, \quad \big\{||x||_2^2 x_i\big\}_{i\in[n]}, \quad \big\{x_i x_j\big\}_{i\neq j}, \quad \big\{x_i^2\big\}_{i\in[n]}, \quad \big\{x_i\big\}_{i\in[n]}, \quad \text{and} \quad 1.$$

(Note here that $||x||_2^2 = \sum_{i\in[n]} x_i^2$.) Altogether there are $1+n+\binom{n}{2}+n+n+1 = \binom{n}{2}+3n+2$ such functions, and so $k$ is at most this number by (a).

7. Let $\mathcal{F}$ be a collection of functions from $[n]$ to $\mathbb{Z}$. Suppose that, for every pair of distinct functions $f, g \in \mathcal{F}$ we have $f(i) = g(i) + 1$ for some $i$. Prove that $|\mathcal{F}| \leq 2^n$. [Hint: look for a suitable collection of polynomials.]

   To each $f \in \mathcal{F}$ associate a polynomial $p_f \in \mathbb{Z}[x_1, \ldots, x_n]$ given by

   $$p_f(x_1, \ldots, x_n) = \prod_{i\in[n]} \big(f(i) + 1 - x_i\big).$$

   Note that the hypothesis gives $p_f\big(g(1), \ldots, g(n)\big) = \mathbf{1}_{f=g}$ for all $f, g \in \mathcal{F}$.

   **Claim:** The polynomials $\{p_f\}_{f\in\mathcal{F}}$ are linearly independent over $\mathbb{Q}$.

   To see this, suppose $\sum_{f\in\mathcal{F}} \lambda_f p_f = 0$, where $\lambda_f \in \mathbb{Q}$ for all $f \in \mathcal{F}$. Then for every $g \in \mathcal{F}$

   $$\lambda_g = \sum_{f\in\mathcal{F}} \lambda_f p_f\big((g(1), \ldots, g(n))\big) = 0.$$

   Thus $\{p_f\}_{f\in\mathcal{F}}$ are linearly independent, proving the claim.

   Lastly, note that all the polynomials $\{p_f\}_{f\in\mathcal{F}}$ are multilinear, and so contained in $V = \text{span}(\{\prod_{i\in I} x_i : I \subset [n]\})$. It follows that $|\mathcal{F}| \leq \dim_{\mathbb{Q}}(V) \leq 2^n$.

8. Let $p$ be a prime and let $n \geq p^2$.

   (a) Prove that if $\mathcal{A} \subset [n]^{(p^2)}$ with $|A \cap B| \not\equiv 0 \mod p$ for all distinct $A, B \in \mathcal{A}$ then $|\mathcal{A}| \leq \binom{n}{\leq p}$.

   (b) Prove that if $\mathcal{A} \subset [n]^{(p^2)}$ with $|A \cap B| \equiv 0 \mod p$ for all $A, B \in \mathcal{A}$ then $|\mathcal{A}| \leq \binom{n}{\leq p}$.

   Using (a) and (b), show that for all $\varepsilon > 0$ there is $N_0$ so that the following holds for $N \geq N_0$. The pairs in $[N]^{(2)}$ can be coloured `red` or `blue` so that $X^{(2)}$ receives both colours for every $X \subset [N]$ with $|X| \geq N^{\varepsilon}$.

   (a) Note that $|A| \equiv 0 \mod p$ for all $A \in \mathcal{A}$. Also $|A \cap B| \not\equiv 0 \mod p$ for all distinct $A, B \in \mathcal{A}$. We may therefore apply the Modular Frankl–Wilson theorem to $\mathcal{A}$, taking $p$ as the prime and $S = \{1, \ldots p-1\}$ to get $|\mathcal{A}| \leq \binom{n}{\leq p}$.

   (b) Here we note that $\mathcal{A}$ is $S$-intersecting, where $S = \{0, p, 2p, \ldots, (p-1)p\}$. By the Frankl–Wilson theorem we obtain that $|\mathcal{A}| \leq \binom{n}{\leq p}$.

For the final part of the question, take $p$ to be a large prime (below we need $p \geq 2/\varepsilon$). We will prove the statement for $N$ of the form $N = n^{p^2}$, where $n \in \mathbb{N}$ is sufficiently large (need $n \geq p^4$ below); this implies the result for all $N$, by an approximation argument.

To begin, identify the elements of $[N]$ with sets in $[n]^{(p^2)}$ in an arbitrary way. Given two distinct sets $A, B \in [n]^{(p^2)}$ colour $AB$ `red` if $A \cap B \not\equiv 0 \mod p$ and `blue` if $A \cap B \equiv 0 \mod p$. This gives a colouring of $[N]^{(2)}$ and note that for any $X \subset [N]$, if $X^{(2)}$ only receives colour `red` then by (a) the set $X$ satisfies $|X| \leq \binom{n}{\leq p}$. The same holds if $X^{(2)}$ only receives colour `blue` by (b). For $n \geq 3p$ we have

$$\binom{n}{\leq p} \leq \frac{n^p}{p!}\Big(1 + (\frac{p}{n-p+1}) + (\frac{p}{n-p+1})^2 + \dots\Big) \leq n^p.$$

Provided $n \geq p^4$ and $p \geq 2/\varepsilon$, this gives

$$|X| \leq \binom{n}{\leq p} \leq n^p \leq \Big(\frac{n}{p^2}\Big)^{2p} \leq \binom{n}{p^2}^{2/p} \leq N^{\varepsilon}.$$

9. Prove that there is an uncountable collection $\mathcal{A}$ of subsets of $\mathbb{N}$ such that $|A \cap B|$ is finite for all distinct $A, B \in \mathcal{A}$.

One possible solution here is as follows. Let $p_i$ denote the $i$th prime for all $i \in \mathbb{N}$. Given $S \subset \mathbb{N}$, consider the set

$$A_S = \Big\{ \prod_{i \in S \cap [n]} p_i : n \in \mathbb{N} \Big\} \subset \mathbb{N}.$$

We claim that $\mathcal{A} = \{A_S : S \subset \mathbb{N}\}$ has the required property.

First note that $A_{S_1} \neq A_{S_2}$ whenever $S_1$ and $S_2$ are distinct, so $\mathcal{A}$ is an uncountable family. Also, if $A_{S_1}$ and $A_{S_2}$ are distinct sets in $\mathcal{A}$ then there is $i \in S_1 \setminus S_2$ (say). All but a finite number of elements of $A_{S_1}$ are divisible by $p_i$, while no element of $A_{S_2}$ is divisible by $p_i$. It follows that $|A_{S_1} \cap A_{S_2}| < \infty$, as required.

10.$^+$ Let $1 \leq i \leq j \leq n$. Let $A = (A_{ST})$ be a $\binom{n}{i} \times \binom{n}{j}$ matrix with rows indexed by elements of $[n]^{(i)}$ and columns indexed by elements of $[n]^{(j)}$, where $a_{ST} = 1$ if $S \subset T$ and $a_{ST} = 0$ otherwise. Prove that $\text{rank}(A) = \min\{\binom{n}{i}, \binom{n}{j}\}$.

Omitted. Feel free to talk to me about this though.