C8.1 STOCHASTIC DIFFERENTIAL EQUATIONS

EXERCISE SHEET 1

(1) Let $\{W_t : t \geq 0\}$ be a standard Brownian motion in $\mathbb{R}^2$, and $D_R = \{z \in \mathbb{R}^2 : |z| < R\}$ be the disk with radius $R > 0$.
  (a) Compute $\mathbb{P}(W_t \in D_R)$ where $t > 0$ and hence show that
    (i) $\mathbb{P}\left(W_t \notin D_{\sqrt{2\lambda t}}\right) = e^{-\lambda}$.
    (ii) Let $|D_R|$ be the Lebesgue measure of the domain D_R. Then
$$\lim_{R\to 0} \frac{\mathbb{P}(W_t \in D_R)}{|D_R|} = \frac{1}{2\pi t}.$$
    (iii) What happens for a Brownian motion in $\mathbb{R}^3$ started at x in (a)(i),(ii) when we replace D_R by $B_R(x)$, the ball of radius R centred at x?

(2) Let W be a standard Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
  (a) Show that the process B defined by $B_t = tW_{1/t}$ is a Brownian motion (you may assume continuity at 0).
  (b) Let $T_a = \inf\{t \geq 0 : W_t = a\}$ denote the first hitting time of a. Show using the reflection principle or otherwise that
$$\mathbb{P}(T_a < t) = 2\mathbb{P}(W_t > a)$$
  and hence the probability density function of T_a is
$$f_{T_a}(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right)$$
  (c) Let $S_a = \sup\{t \geq 0 : W_t = at\}$. Is S_a a stopping time? Show that $S_a = \frac{1}{T_a}$ in distribution and hence find $\mathbb{E}S_a$ and $\mathbb{E}W_{S_a}$.

(3) A real-valued process is centred Gaussian, if its finite-dimensional distributions are normal distributions with mean zero. A centred Gaussian process $X = (X_t)_{t\geq 0}$ is called a fractional Brownian motion (FBM) with Hurst parameter $h \in (0,1)$ if $\mathbb{P}(X_0 = 0) = 1$ and its covariance function
$$\mathbb{E}[X_t X_s] = \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right).$$
  (a) Show that for $t > s$, $X_t - X_s$ has mean zero and variance $|t-s|^{2h}$.
  (b) Show that for any $t > s$ and $p > 0$
$$\mathbb{E}[|X_t - X_s|^p] = c_p |t-s|^{hp}$$
  for some constant c_p depending only on p.
  (c) Show that the fractional Brownian motion X has a continuous modification and determine its Holder exponent. [You may wish to recall Kolmogorov's Continuity Theorem]
  (d) By considering a partition with a mesh size that goes to 0, find the value of p such that the p-variation of X is finite and non-zero. Is the process X a semi-martingale for any h?

(4) Let M be a continuous local martingale with $M_0 = 0$. Show that
  (a) M is L^2-bounded if $\mathbb{E}\langle M\rangle_\infty < \infty$ and in this case $M^2 - \langle M\rangle$ is uniformly integrable.
  (b) M converges almost surely as $t \to \infty$ on the set where $\langle M\rangle_\infty < \infty$.