C8.1 STOCHASTIC DIFFERENTIAL EQUATIONS

EXERCISE SHEET 2

(1) Let $W = \left(W_t^1, \dots, W_t^d\right)_{t \geq 0}$ be a standard BM in $\mathbb{R}^d$and $F = \left(F_t^1, \dots, F_t^d\right)_{t \geq 0}$ an adapted continuous stochastic process such that

$$\int_0^t \left|F_s^i\right|^2 ds < \infty$$

for $i = 1, \dots, d$ and $t \geq 0$. Show that

(a) for any $i, j \leq d$ $\left\langle B^i, B^j \right\rangle_t = \delta_{ij} t$ where $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$

(b) for any $\left\langle \int_0^\cdot F_s^i dW_s^i, \int_0^\cdot F_s^j dW_s^j \right\rangle_t = \delta_{ij} \int_0^t F_s^i F_s^j ds$

(c) $X_t = \left(\sum_{i=1}^d F_s^i dW_s^i\right)^2 - \sum_{i=1}^d \int_0^t \left|F_s^i\right|^2 ds$ is a martingale

(d) for any $\lambda > 0$ $\mathbb{P}\left(\sup_{0 \leq s \leq t} \left|\sum_{i=1}^d \int_0^s F_s^i dW_s^i\right| \geq \lambda\right) \leq \lambda^{-2} \int_0^t \mathbb{E}\left[\sum_{i=1}^d \left|F^i\right|^2\right] ds$

(2) Let X be a positive random variable, independent of a standard Brownian motion W on a filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t), \mathbb{P})$. Let $M = W_{tX}$ and assume that the filtration $(\mathscr{F}_t) := (\mathscr{G}_{tX})$ is such that X is $\mathscr{F}_t$-measurable for all $t \geq 0$.

(a) Show that M is a local martingale wrt to $(\mathscr{F}_t)$.

(b) Show that M is a martingale if and only if $\mathbb{E}\left[\sqrt{X}\right] < \infty$.

(c) Calculate $\langle M \rangle_t$.

(d) Let A be an increasing process, independent of W and $A_0 = 0$. Assuming it is adapted to a filtration $\left(\mathscr{F}_t^A\right) := (\mathscr{G}_{A_t})$, show that $(W_{A_t})_{t \geq 0}$ is a local martingale with respect to $\left(\mathscr{F}_t^A\right)$ and find conditions which ensure it is a martingale and determine its quadratic variation.

(3) Let W be a standard Brownian motion. Find the SDEs satisfied by the following processes and determine which are martingales.

(a) $X_t = e^{t/2} \cos W_t$

(b) $X_t = t W_t$

(c) $X_t = (W_t + t) \exp\left(-W_t - \frac{t}{2}\right)$

(d) $X_t = \left(W_t^1\right)^2 + \left(W_t^2\right)^2$ where $\left(W^1, W^2\right)$ is a two-dimensional Brownian motion

(4) Let $W = \left(W_t^1, \dots, W_t^d\right)_{t \geq 0}$ be a standard Brownian motion in $\mathbb{R}^d$. Let $X_t = \|W_t\| = \sqrt{\left(W_t^1\right)^2 + \dots + \left(W_t^d\right)^2}$.

(a) Find the SDE for X. Show that

$$X_t = X_0 + \int \frac{d-1}{2X_s} ds + B_t$$

where B is a Brownian motion in $\mathbb{R}$.

(b) For $\beta_k(t) = \mathbb{E}\left[X_t^{2k}\right]$ and $k \geq 0$ show that

$$\beta_k(t) = k(2(k-1) + d) \int_0^t \beta_{k-1}(s) ds.$$

(c) Calculate $\mathbb{E}\left[\|W_t\|^4\right] = \mathbb{E}\left[\|W_t\|^6\right]$.

(5) Let $W = \left(W^1, W^2, W^3\right)$ be a Brownian motion in $\mathbb{R}^3$, where W_0 is a random variable in $\mathbb{R}^3 \setminus \{0\}$, independent of $(W_t - W_0)_{t > 0}$.

(a) Define $\|.\|$ as in question 4, show that $\|W\|^{-1}$ is a local martingale

(b) Suppose $W_0 = y$. Let $M_t = \|W_{t+1} - y\|^{-1}$, for $t \geq 0$. Show by a direct calculation that $\mathbb{E}\left[M_t^2\right] = \frac{1}{1+t}$. Deduce that M is bounded in L^2 and uniformly integrable. You may assume that $\mathbb{P}(\forall t > 0, W_{t+1} = y) = 0$.

(c) Show that M is both a local martingale and a supermartingale.

(d) Use the martingale convergence theorem to show that M is not a martingale

(6) Let B be a standard real-valued Brownian motion. Prove that

$$X_t = \int_0^t \operatorname{sgn}(B_s - x)\, dB_s$$

is a Brownian motion, where $x \in \mathbb{R}$ and

$$\operatorname{sgn} = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(Hint: use Levy's characterization of Brownian motion).

(7) Let $B = \left(B_t^1, B_t^2\right)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}^2$. Prove that the process $X = \left(X^1, X^2\right)$ defined as

$$\begin{aligned} X_t^1 &= \int_0^t \cos\left(B_s^1\right) dB_s^1 - \sin\left(B_s^1\right) dB_s^2 \\ X_t^1 &= \int_0^t \sin\left(B_s^1\right) dB_s^1 + \cos\left(B_s^1\right) dB_s^2 \end{aligned}$$

is again a standard Brownian motion in $\mathbb{R}^2$.

(8) Let X,Y be continuous semimartingales. Define the stochastic exponential

$$\mathscr{E}(X)_t := \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right)$$

and prove that there exists a unique continuous semimartingale Z such that

$$Z_t = Y_t + \int_0^t Z_s dX_s$$

and that Z is given by

$$Z_t = \mathscr{E}(X)_t \left(Y_0 + \int_0^t \mathscr{E}(X)_s^{-1} dY_s - \int_0^t \mathscr{E}(X)_s^{-1} d\langle X, Y\rangle_s\right).$$