C8.1 STOCHASTIC DIFFERENTIAL EQUATIONS

EXERCISE SHEET 3

(1) Let $M \in \mathcal{M}_{\text{c,loc}}$.
   (a) Show that the intervals of constancy of $t \mapsto M_t$ and $t \mapsto \langle M \rangle_t$ coincide a.s.
   (b) Show that if

$$\mathbb{E}\left[\exp\left(iu\left(M_t - M_s\right)\right)|\mathcal{F}_s\right] = \exp\left(-\frac{u^2\left(t-s\right)}{2}\right) \text{ for all } s < t$$

   then M is a Brownian motion.

(2) Show that Novikov's criterion implies the Kazamaki criterion: if $M \in \mathcal{M}_{\text{c,loc}}$ and

$$\mathbb{E}\left[\exp\frac{1}{2}\langle M \rangle_\infty\right] < \infty \tag{0.1}$$

then $\exp\left(\frac{1}{2}M\right)$ is a u.i. submartingale. Hint: use BDG to show that M is a u.i. martingale, then argue that this implies that $\exp\left(\frac{1}{2}M\right)$ is a u.i. submartingale

(3) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions and $\mathbb{Q} \sim \mathbb{P}$ an equivalent probability measure ($\mathbb{P}$ is absolutely continuous wrt to $\mathbb{Q}$ and $\mathbb{Q}$ is absolutely continuous wrt to $\mathbb{P}$). Denote the density with

$$\rho_\infty := \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ and define } \rho_t := \mathbb{E}_{\mathbb{P}}\left[\rho_\infty|\mathcal{F}_t\right] \text{ for } t \geq 0.$$

Show that
   (a) $\mathbb{E}_{\mathbb{Q}}\left[Z|\mathcal{F}_t\right] = \frac{\mathbb{E}_{\mathbb{P}}[\rho_\infty Z|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{P}}[\rho_\infty|\mathcal{F}_t]}$ for any real-valued, bounded random variable Z,
   (b) $\rho = (\rho_t)_{t\geq 0}$ is a u.i. $((\mathcal{F}_t), \mathbb{P})$-martingale,
   (c) $\rho^{-1} = \left(\frac{1}{\rho_t}\right)_{t\geq 0}$ is a u.i. $((\mathcal{F}_t), \mathbb{Q})$-martingale and $\frac{1}{\rho_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\rho_\infty}|\mathcal{F}_t\right]$.

(4) Let $M \in \mathcal{M}_{\text{c,loc}}$ with $M_0 = 0$. Show that
   (a) $\mathcal{E}(M) \in \mathcal{M}_{\text{c,loc}}$, $\mathcal{E}(M)_t \geq 0$ for all $t \geq 0$,
   (b) $\mathcal{E}(M)$ is a supermartingale with $\mathbb{E}[\mathcal{E}(M)_t] \leq 1$,
   (c) $\mathcal{E}(M) \in \mathcal{M}_{\text{c}}$ iff $\mathbb{E}[\mathcal{E}(M)_t] = 1$ for all $t \geq 0$.

(5) Let B be a standard real-valued Brownian motion. Prove that

$$B_t^4 = 3t^2 + \int_0^t \left(12\left(t-s\right)B_s + 4B_s^3\right) dB_s.$$

(6) Let B be a standard real-valued Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, M a continuous $(\sigma(B), \mathbb{P})$-martingale and with $\sup_t |M_t|_{L^2(d\mathbb{P})} < \infty$. Assume there exists a predictable process H such that

$$M_t = M_0 + \int_0^t H_s dB_s.$$

Show that such an H must be unique.

(7) Prove Kazamaki's condition: if $M \in \mathcal{M}_{\text{c,loc}}$ with $M_0 = 0$ and such that

$$\left(\exp\frac{1}{2}M_t\right)_{t\geq 0}$$

is a u.i. submartingale then $\mathcal{E}(M) \in \mathcal{M}_c$.

(a) For $\alpha \in (0,1)$ show that

$$\mathcal{E}(\alpha M)_t = (\mathcal{E}(M)_t)^{\alpha^2} \left(Z_t^\alpha\right)^{1-\alpha^2}$$

with $Z_t^\alpha := \exp\left(\frac{\alpha}{1+\alpha} M_t\right)$.

(b) Show that the family of random variables

$$\{\mathcal{E}(\alpha M)_\tau : \tau \text{ a stopping time}\}$$

is u.i. and that $\mathcal{E}(\alpha M)$ is a u.i. martingale

(c) Use the existence of M_∞ and the dominated convergence theorem to show that $\mathcal{E}(M)$ is a martingale.

(8) Let B be a real-valued Brownian motion. Prove that $X = (X_t)_{t\geq 0}$ defined as

$$X_t = \exp\left(\int_0^t \sqrt{s}\sin(B_s)\,dB_s - \frac{1}{2}\int_0^t s\sin^2(B_s)\,ds\right)$$

is a continuous martingale.