C8.1 STOCHASTIC DIFFERENTIAL EQUATIONS

EXERCISE SHEET 4

(1) We call a function $F:[0,\infty)\to\mathbb{R}$ moderately increasing with growth rate k if F is nondecreasing, $F(0)=0$ and

$$F(2\lambda)\le kF(\lambda).$$

Show that if X,Y are two nonnegative random variables that fulfill $\forall\delta,\lambda>0$

$$\mathbb{P}(X>2\lambda, Y\le\delta\lambda)\le\delta^2\mathbb{P}(X>\lambda)$$

then for every moderately increasing function F, there exists a constanct $c=c(k)$ that depends only on the growth rate k of F such that

$$\mathbb{E}[F(X)]\le c\mathbb{E}[F(Y)].$$

(a) Wlog F can be assumed to be bounded,
(b) Show that $\mathbb{E}[F(Z)]=\int_0^\infty\mathbb{P}(Z>\lambda)\,dF(\lambda)$ for any nonnegative random variable Z
(c) Conclude by replacing Z with $\frac{X}{2}$ and $\frac{Y}{\delta}$.

(2) Let $\beta>1$. Show that $\forall\delta,\lambda>0$

$$\begin{aligned}\mathbb{P}\left(B^\star_\tau>\beta\lambda,\sqrt{\tau}\le\delta\lambda\right) &\le \frac{\delta^2}{(\beta-1)^2}\mathbb{P}\left(B^\star_\tau>\lambda\right),\\ \mathbb{P}\left(\sqrt{\tau}>\beta\lambda,B^\star_\tau\le\delta\lambda\right) &\le \frac{\delta^2}{(\beta^2-1)}\mathbb{P}\left(\sqrt{\tau}>\lambda\right).\end{aligned}$$

(a) Relate the stopping times

$$\sigma_1=\inf\{t:|B_{t\wedge\tau}|>\lambda\},\sigma_2=\inf\{t:|B_{t\wedge\tau}|>\beta\lambda\},\sigma_3=\inf\left\{t:\sqrt{t\wedge\tau}>\delta\lambda\right\}$$

and the events $\{\sigma_1<\sigma_2<\tau\}$ and $\{\sigma_3=\infty\}$ to the events $\left\{B^\star_\tau>\beta\lambda\right\}$ and $\left\{\sqrt{\tau}\le\delta\lambda\right\}$,
(b) Show that $\mathbb{E}\left[(B_\sigma-B_{\sigma'})^2\right]=\mathbb{E}[\sigma-\sigma']$ for all bounded stopping times $\sigma\ge\sigma'$,
(c) Use Chebyshev's inequality and (2b), (2a) to derive the first inequality,
(d) For the second inequality interchange the roles of $B_{\tau\wedge t}$ and $(t\wedge\tau)$ and give a similar argument

(3) Consider the SDE $\mathrm{e}_x(x\mapsto\mu x, x\mapsto\sigma))$ with $\mu\in\mathbb{R}^{e\times e}$,$\sigma\in\mathbb{R}^{e\times d}$ constant.
(a) Denote with $\exp(M):=\sum_{k\ge0}\frac{M^k}{k!}$ the matrix exponential and use the process $(\exp(t\mu)X_t)_{t\ge0}$ to derive an explicit solution (X,B).
(b) What is the distribution of X_t for $t>0$?
(c) Let $e=d=1$. Show that for any bounded measurable, real-valued function f

$$\mathbb{E}[f(X_t)]=\mathbb{E}'\left[f\left(e^{t\mu}x+N\sqrt{\frac{\sigma^2}{2}\left(e^{2t\mu}-1\right)}\right)\right]$$

where the expectation $\mathbb{E}'$ is taken over any probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ that carries a standard Gausian, $N\sim\mathcal{N}(0,1)$.
(d) Find a function $p(t,x,y)$ in explicit form such that $\mathbb{E}[f(X_t)]=\int_{\mathbb{R}^e}f(y)p(t,x,y)\,dy$.

(4) Consider the SDE $\mathrm{e}_x(x\mapsto\mu x, x\mapsto\sigma x)$ with $\mu,\sigma\in\mathbb{R}$.
(a) Find an explicit solution (X,B),

(b) Show that for any bounded measurable, real-valued function f

$$\mathbb{E}[f(X_t)] = \mathbb{E}'\left[f\left(xe^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}N}\right)\right]$$

where the expectation $\mathbb{E}'$ is taken over any probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ that carries a standard Gausian, $N \sim \mathcal{N}(0,1)$.

(5) For $\alpha > 0$ define $\sigma_\alpha : \mathbb{R}^2 \to \mathbb{R}$ as $\sigma_\alpha(x) = |x|^\alpha \wedge 1$.
- (a) Find a trivial solution for SDE $\mathrm{e}_0(0, \sigma_\alpha)$ for all $\alpha > 0$.
- (b) Show that if $\alpha \geq 1$ this is a strong solution.
- (c) If $\alpha < 1$ show that one can find a time change of the Brownian motion to give another solution:
  - (i) If $\left((X^i)_{i=1,2}, B\right)$ is such a solution, show that $\langle X^i \rangle$ does not depend on i and conclude that for
  $$\tau_t := \inf\left\{s : \langle X^i \rangle_s > t\right\}$$
  we have $\tau_t = \int_0^t \sigma_\alpha^{-2}(B_s)\,ds$
  - (ii) Compute $\mathbb{E}[\tau_t]$ and conclude that the time change $t \mapsto \tau_t$ is finite a.s. Use Dambis–Dubins–Schwarz to construct another solution.

(6) Fix $\lambda, \nu, \sigma > 0$ and let $\mu(x) = \lambda(\nu - x)$.
- (a) Find a an explicit solution (X, B) of $\mathrm{e}_x(\mu, x \mapsto \sigma)$ (Hint: use Ito with $(t,x) \mapsto xe^{\lambda t}$).
- (b) Calculate mean and variance of X_t for $t > 0$.
- (c) Let $\nu = 0$. Show that there exists a Brownian motion W such that (Y, X) with $Y := X^2$ is a solution of the SDE $e_{x^2}(\mu', \sigma')$ with $\mu(y) = -2\lambda y + \sigma^2$,$\sigma(y) = 2\sigma\sqrt{y}$.

(7)
- (a) Prove Skorokhod's lemma that was stated in the lecture (and lecture notes) and deduce that if B is a Brownian motion, the process $X_t := \int_0^t \operatorname{sgn}(B_s)\,dB_s$ is a also Brownian motion and the filtration generated by X is the same as the filtration generated by $|B|$. Moreover,

$$L_t^0(B) = -\sup_{s\leq t} X_s \text{ for all } t \geq 0, \text{a.s.} \tag{1.1}$$

- (b) Let $S_t := \sup_{s\leq t} B_s$. Show that the two-dimensional processes $\left(|B|, L^0(B)\right)$ and $(S - B, S)$ have the same distribution.
- (c) Show that the SDE $\mathrm{e}_0(0, \sigma)$ with $\sigma(x) := \operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0. \end{cases}$ has a weak solution but no strong solution.
- (d) Show that for the SDE $\mathrm{e}_0(0, \sigma)$ uniqueness in law holds but pathwise uniqueness does not hold.

(8) Let B be a standard Brownian motion. Show that

$$Z_t := \mathcal{E}(-\alpha B)_t$$

is the solution of an ordinary differential equation with a random vector field. Use this to find a process Y that solves the SDE

$$dY_t = \lambda dt + \alpha Y_t dB_t \text{ and } X_0 = x \tag{1.2}$$

for fixed $\lambda, \alpha, x \in \mathbb{R}$.