C7.5: General Relativity I
PS4 Answers

1. Kepler's 3rd law

What is the physical meaning of the coordinate time t in the Schwarzschild solution?

The coordinate time t is the proper time of a stationary observer at spatial infinity ($r \to \infty$).

Show that the proper time τ of an observer on a circular orbit at radius R and the coordinate time t are related by

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1 - 3M/R}.$$

The observer follows a circular geodesic, so $\mathcal{L}$, E and J are conserved and the geodesic equations are satisfied with $\dot{r} = \ddot{r} = 0$. The geodesic is timelike so $\mathcal{L} = -1$ which gives

$$-1 = -f\dot{t}^2 + f^{-1}\dot{r}^2 + r^2\dot{\phi}^2, \tag{1}$$

where $f = 1 - 2M/r$. The r equation of motion is

$$2f^{-1}\ddot{r} - 4Mr^{-2}f^{-2}\dot{r} = -2Mr^{-2}\dot{t}^2 - 2Mr^{-2}f^{-2}\dot{r}^2 + 2r\dot{\phi}^2.$$

As the orbit is circular, this simplifies to

$$r\dot{\phi}^2 = Mr^{-2}\dot{t}^2. \tag{2}$$

Substituting this into (1) and setting $r = R$ and $\dot{r} = 0$ gives

$$\begin{aligned} -1 &= -f\dot{t}^2 + MR^{-1}\dot{t}^2 \\ &= -\dot{t}^2 + 2MR^{-1}\dot{t}^2 + MR^{-1}\dot{t}^2 \\ &= -\dot{t}^2 + 3MR^{-1}\dot{t}^2, \end{aligned}$$

and rearranging we have

$$\dot{t}^2 = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = \frac{1}{1 - 3M/R}.$$

Using this result, show that

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{M}{R^3},$$

for a circular orbit at radius R.

Simply rearrange (2) to get

$$\frac{\dot{\phi}^2}{\dot{t}^2} = \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{M}{R^3}.$$

2. Stability of circular orbits

Consider a time-like geodesic in the Schwarzschild geometry that is a small perturbation from a circular orbit at radius R in the equatorial plane $\theta = \pi/2$

$$r(\tau) = R + \varepsilon(\tau).$$

Show that the perturbation must solve an equation of the form

$$\ddot{\varepsilon} + f(R)\varepsilon = 0,$$

and find the function $f(R)$.

Start from the orbit equation for a timelike geodesic

$$\begin{aligned} \frac{E^2-1}{2} &= \frac{1}{2}\dot{r}^2 + V(r), \\ V(r) &= -\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}. \end{aligned}$$

Using this or the r equation of motion, upon taking an r derivative we have

$$\begin{aligned} \ddot{r} &= -\frac{\partial V}{\partial r} \\ &= -\frac{M}{r^2} + \frac{J^2}{r^3} - 3\frac{MJ^2}{r^4}. \end{aligned}$$

Let $r = R + \varepsilon$ and work to order $\mathcal{O}(\varepsilon)$. We then have

$$\begin{aligned} \ddot{\varepsilon} &= -\frac{1}{R^2}\frac{M}{(1+\varepsilon/R)^2} + \frac{1}{R^3}\frac{J^2}{(1+\varepsilon/R)^3} - \frac{3}{R^4}\frac{MJ^2}{(1+\varepsilon/R)^4} \\ &\simeq -\frac{M}{R^2}(1-2\varepsilon/R) + \frac{J^2}{R^3}(1-3\varepsilon/R) - \frac{3MJ^2}{R^4}(1-4\varepsilon/R). \end{aligned}$$

The terms at zeroth order in ε vanish for a circular orbit (as $\ddot{r} = 0$ for the unperturbed solution). The remaining terms give

$$\ddot{\varepsilon} = \left(\frac{2M}{R^3} - \frac{3J^2}{R^4} + \frac{12MJ^2}{R^5}\right)\varepsilon.$$

From the previous question, for a circular orbit we have

$$\dot{t}^2 = (1 - 3M/R)^{-1}, \qquad \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{M}{R^3},$$

which together give

$$\dot{\phi}^2 = \frac{M}{R^3}(1 - 3M/R)^{-1},$$

so that J^2 is given by

$$J^2 = R^4\dot{\phi}^2 = MR(1 - 3M/R)^{-1}.$$

Using this in the equation for ε gives

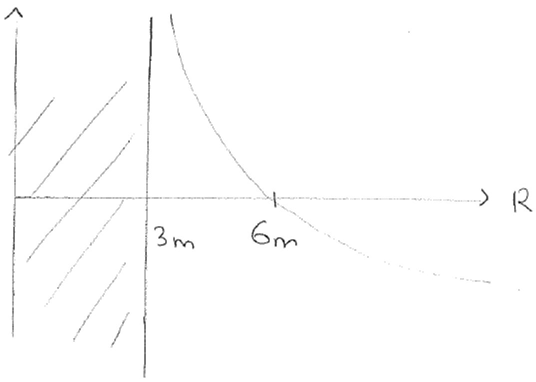
$$\ddot{\varepsilon} = -\frac{M}{R^3}\frac{1 - 6M/R}{1 - 3M/R}\varepsilon.$$

Thus the function $f(R)$ is

$$f(R) = \frac{M}{R^3}\frac{1 - 6M/R}{1 - 3M/R}.$$

Plot this function, showing clearly its asymptote and intercept. Hence, re-derive the fact that circular orbits only exist if $R > 3M$ and show that these orbits are stable if $R > 6M$.

The function $f(R)$ looks like



We just found that

$$J^2 = R^4\dot{\phi}^2 = MR(1 - 3M/R)^{-1},$$

so J is real and finite only for $R > 3M$. Circular orbits can exist only for $R > 3M$. For these orbits to be stable, small perturbations must remain small for all time. The equation $\ddot{\varepsilon} + f(R)\varepsilon = 0$ has oscillatory (stable) solutions for $f(R) > 0$ and exponential (unstable) solutions for $f(R) < 0$. Thus we see stable solutions exist only for $R > 6M$.

3. Meaning of E

Consider a stationary observer at radius R in the Schwarzschild geometry and a massive test particle moving on a time-like geodesic $x^a(\tau)$ that intersect at some point P. Show that the stationary observer measures the energy per-unit-rest mass of the test particle to be

$$\sqrt{1-\frac{2M}{R}}\,\dot{t}.$$

A stationary observer $\mathcal{O}$ has 4-velocity $u^a = (u^0, 0, 0, 0)$ with

$$g_{ab}u^a u^b = -1 \quad \Rightarrow \quad u^0 = (1-2M/R)^{-1/2}.$$

The test particle has 4-velocity $\dot{x}^a = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ in this frame. The observer $\mathcal{O}$ measures the energy per unit rest mass of the test particle at P to be

$$\begin{aligned}\mathcal{E}_{\mathcal{O}} &= -g_{ab}u^a\dot{x}^b|_P \\ &= (1-2M/R)u^0\dot{x}^0 \\ &= (1-2M/R)^{1/2}\dot{t}.\end{aligned}$$

Let the test particle and the stationary observer have relative velocity v at the point P. Explain why

$$\gamma(v) = \sqrt{1-\frac{2M}{R}}\,\dot{t}.$$

$\mathcal{E}_{\mathcal{O}}$ is invariant and does not depend on the frame we use. In particular, we can go to a *local* inertial frame at P in which the observer $\mathcal{O}$ is stationary ($u'^{\mu} = (1, \mathbf{0})$) and the test particle is moving with velocity $\boldsymbol{v}$ ($\dot{x}'^{\mu} = \gamma(\boldsymbol{v})(1, \boldsymbol{v})$). In this frame we have

$$\begin{aligned}\mathcal{E}_{\mathcal{O}} &= -\eta_{\mu\nu}u'^{\mu}\dot{x}'^{\nu}|_P \\ &= \gamma(\boldsymbol{v}).\end{aligned}$$

Thus we have

$$\gamma(\boldsymbol{v}) = \sqrt{1-\frac{2M}{R}}\,\dot{t}.$$

Now derive an expression for the conserved quantity E. Expanding this expression for large distances ($R \gg 2M$) and small velocities ($v \ll 1$), show that E is approximately the sum of the rest mass, kinetic energy and potential energy.

The conserved quantity E is

$$\begin{aligned}E &= (1-2M/R)\dot{t} \\ &= (1-2M/R)^{1/2}\gamma(\boldsymbol{v}) \\ &= (1-2M/R)^{1/2}(1-\boldsymbol{v}^2)^{-1/2}.\end{aligned}$$

We then expand for large distance and small velocity to give

$$
\begin{aligned}
E &= (1 - M/R + \ldots)(1 + \tfrac{1}{2}\boldsymbol{v}^2 + \ldots) \\
&\simeq 1 + \tfrac{1}{2}\boldsymbol{v}^2 - \frac{M}{R}.
\end{aligned}
$$

This is a sum of the rest mass, kinetic energy and potential energy of the particle per unit rest mass.

This is an approximate characterization of the conserved quantity E. To find the precise meaning, suppose the stationary observer converts the energy measured into a photon and sends it out to another stationary observer at $r \to \infty$. What is the energy of the photon measured by a stationary observer at infinity?

Consider a second observer at spatial infinity who measures the energy of the photon to be $\mathcal{E}_\infty$. This is related to $\mathcal{E}_\mathcal{O}$ by the usual gravitational redshift as

$$
\begin{aligned}
\mathcal{E}_\infty &= \mathcal{E}_\mathcal{O} \times \sqrt{1 - \frac{2M}{R}} \\
&= \sqrt{1 - \frac{2M}{R}}\,\dot{t} \times \sqrt{1 - \frac{2M}{R}} \\
&= \left(1 - \frac{2M}{R}\right)\dot{t} \\
&= E.
\end{aligned}
$$

So the conserved quantity E is the energy of the test particle as measured by a stationary observed at spatial infinity.

4. Acceleration

Observers not freely falling experience acceleration forces. This is encoded in the acceleration 4-vector $A^a = V^b \nabla_b V^a$, where V^a is the 4-velocity. This measures the failure of the corresponding curve to be a geodesic.

Consider a stationary observer at radius R in the Schwarzschild geometry and show that their acceleration four-vector is

$$
A^a = (0, m/R^2, 0, 0).
$$

You will need to compute the Christoffel symbol $\Gamma^r{}_{tt}$ for the Schwarzschild metric from the r equation of motion.

A stationary observer has 4-velocity

$$
V^a = \big((1 - 2M/r)^{-1/2}, 0, 0, 0\big).
$$

The 4-acceleration is

$$\begin{aligned} A^a &= V^b \nabla_b V^a \\ &= V^t \nabla_t V^a \\ &= V^t (\partial_t V^a + \Gamma^a{}_{tt} V^t) \\ &= (V^t)^2 \Gamma^a{}_{tt}. \end{aligned}$$

Only the $a = r$ component is non-zero and this can be read off from the r equation of motion as

$$\Gamma^r{}_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right).$$

This gives

$$A^r = \frac{M}{r^2},$$

with zero for the other components.

Now compute the proper acceleration $a = (g_{ab}A^a A^b)^{1/2}$. Which of these results is more physical?

The proper acceleration is

$$\begin{aligned} a &= (g_{ab}A^a A^b)^{1/2} \\ &= \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2}. \end{aligned}$$

We can relate this to the acceleration measured by the stationary observer in their own local inertial frame as follows. In the rest frame of the stationary observer, its 4-acceleration is given by

$$A'^{\mu} = (0, a', 0, 0),$$

where the acceleration measured is in the radial direction. This is what we might think of as the physical acceleration. Now note that there is an invariant which isolates a' as

$$\begin{aligned} -\eta_{\mu\nu} A'^{\mu} A'^{\nu} &= -g_{ab} A^a A^b \\ &\Downarrow \\ a'^2 &= \left(1 - \frac{2M}{r}\right)^{-1} \frac{M^2}{r^4} \\ &= a^2, \end{aligned}$$

so that $a = a'$. Note also that we must have $A'^{\mu} = (0, a', 0, 0)$ with a positive sign due to the equivalence principle. A stationary observer at r is in a gravitational field acting radially inwards, so to remain at a fixed value of r the observer measures that they are being accelerated radially outwards!

Show that this agrees with the Newtonian expectation for $r \gg 2M$ and that stationary observers can only exist for $R > 2M$.

For $r \gg 2M$, to leading order in $2M/r$ we have

$$\begin{aligned} a &= \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2} \\ &\simeq \left(1 + \frac{M}{r}\right) \frac{M}{r} \frac{1}{r} \\ &= \frac{M}{r^2}, \end{aligned}$$

which is the usual Newtonian result.

For $r \to 2M$ from above, we have $a \to \infty$. As the observer approaches the horizon, it measures an infinite acceleration in its own rest frame. There are no stationary observers for $r \leq 2M$.

5. Capture by a black hole

For geodesics in the Schwarzschild solution

$$\frac{E^2 - \kappa}{2} = \frac{1}{2}\dot{r}^2 + V(r),$$

where

$$V(r) = -\frac{\kappa M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3},$$

with $\kappa = 0$ for null geodesics and $\kappa = 1$ for time-like geodesics

In this question we are interested in when incoming geodesics will be captured by a black hole. For such problems it is convenient to define the *impact parameter* b by

$$b := \frac{J}{\sqrt{E^2 - \kappa}}.$$

a) Massless particle

First consider an incoming null geodesic. Show that a massless particle is captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the *capture cross-section* $\sigma := \pi b_\mathrm{c}^2$ is

$$\sigma = 27\pi M^2.$$

A particle will be captured if it has sufficient energy such that $\dot{r}^2 > 0$ at the maximum of the potential. Given that $\dot{r} < 0$ for an incoming particle, this means it will pass the potential barrier and head towards $r = 0$.

The maximum of the potential is at

$$0 = V'(r) = -\frac{J^2}{r^3} + 3\frac{MJ^2}{r^4}$$
$$\Downarrow$$
$$r_{\text{max}} = 3M.$$

There will be capture if $E^2/2 > V(r_{\text{max}})$. The value of V at the maximum is

$$V(r_{\text{max}}) = \frac{J^2}{54M^2}.$$

There will be capture if $E^2 > J^2/27M^2$. Rewriting this in terms of the impact parameter $b = J/E$ we have

$$b^2 < b_{\text{crit}}^2 = 27M^2.$$

The capture cross-section is

$$\sigma = \pi b_{\text{crit}}^2 = 27\pi M^2.$$

b) Non-relativistic massive particle

Now consider an incoming time-like geodesic. We will assume that the massive particle starts at $r = \infty$ with non-relativistic velocity $v \ll 1$ measured by a stationary observer. Explain why

$$b = \frac{J}{v} + \mathcal{O}(v),$$

and draw a diagram explaining the physical significance of the impact parameter in this case.

Recall from question 3 that the conserved quantity E is given by

$$E \simeq 1 + \tfrac{1}{2}v^2 - \frac{M}{r},$$

for $v \ll 1$ and $r \gg 2M$, where v is the velocity measured by a stationary observer at radius r. Consider an incoming particle with velocity $v \ll 1$ as measured by a stationary observer at spatial infinity. We then have

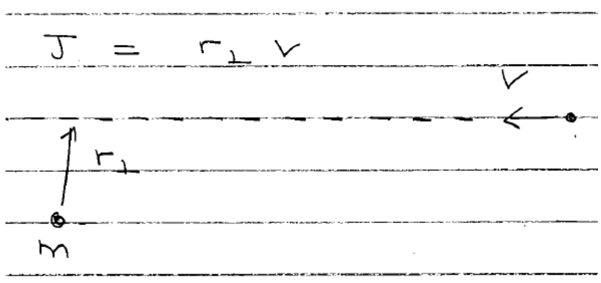
$$E \simeq 1 + \tfrac{1}{2}v^2 + \mathcal{O}(v^4)$$
$$\Downarrow$$
$$\sqrt{E^2 - 1} \simeq v + \mathcal{O}(v^3).$$

The impact parameter is then

$$\begin{aligned} b &= \frac{J}{\sqrt{E^2 - 1}} \\ &= \frac{J}{v} + \mathcal{O}(v). \end{aligned}$$

In the limit $v \ll 1$, the proper time of the particle at spatial infinity reduces to the coordinate time t. Thus J reduces to the usual classical angular momentum per unit rest mass with b the impact parameter:

$$J = r_\perp v, \qquad b = r_\perp.$$



Show that the massive particle will be captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the capture cross-section $\sigma := \pi b_\mathrm{c}^2$ is approximately

$$\sigma = 16\pi M^2/v^2.$$

For capture of a massive particle we want

$$\frac{E^2 - 1}{2} > V(r_\mathrm{max}),$$

so that there is still some inwards radial velocity at the maximum of the potential. This is difficult to do in general, so we use the non-relativistic approximation and $E^2 - 1 \sim \mathcal{O}(v^2)$. The capture condition is then $V(r_\mathrm{max}) < 0$ to order v. Given the graph of $V(r)$ has a single maximum at r_max where $V(r_\mathrm{max})$ approaches zero, we can simply require that $V(r)$ has no real root (and a single real root as $V(r_\mathrm{max}) \to 0$). We have

$$\begin{aligned} V(r) &= -\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3} \\ &= \frac{1}{r^3}\left(-r^2 M + r\frac{J^2}{2} - MJ^2\right). \end{aligned}$$

For a single real root, the descriminant of this equation must be zero ("$b^2 - 4ac = 0$"). We have

$$\begin{aligned} 0 &= \frac{J^4}{4} - 4(-M)(-MJ^2) \\ &= \frac{J^4}{4} - 4M^2J^2 \\ &\Downarrow \\ J &= 4M. \end{aligned}$$

For $V(r_\mathrm{max}) < 0$ we then have $J < 4M$. If this holds, the particle will be captured. Using this, the critical impact parameter is then

$$b_{\text{crit}} = \frac{4M}{v}.$$

The particle will be captured if $b < b_{\text{crit}}$. The capture cross-section is

$$\sigma = \pi b_{\text{crit}}^2 = 16\pi \frac{M^2}{v^2}.$$

6. Einstein equations in cosmology

The FRW metric in coordinates (τ, r, θ, ϕ) is

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \mathrm{d}\Sigma_{(3)}^2,$$
$$\mathrm{d}\Sigma_{(3)}^2 = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2$$

Using this metric, show that the Einstein equations in the presence of a perfect fluid imply

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}, \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P).$$

A perfect fluid with 4-velocity u^a has stress tensor

$$T_{ab} = (p + \rho) u_a u_b + p g_{ab},$$

where $u^a = (1, 0, 0, 0)$ in a comoving coordinate system. Let the FRW metric be

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \tilde{g}_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

Here $\tilde{g}_{ij}$ is maximally symmetric (and x^i are not the usual Cartesian coordinates unless we have $k = 0$). Using the geodesic equation, one finds the non-vanishing components of the Christoffel symbols are

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^i{}_{jk}, \qquad \Gamma^i{}_{j0} = \frac{a'}{a}\delta^i_j, \qquad \Gamma^0{}_{ij} = a' a \tilde{g}_{ij}.$$

The non-zero components of the Riemann tensor are then

$$R_{j0}{}^i{}_0 = -\frac{a''}{a}\delta^i_j, \quad R_{lj}{}^k{}_i = \tilde{R}_{lj}{}^k{}_i + a'^2(\delta^k_l \tilde{g}_{ij} - \delta^k_j \tilde{g}_{il}),$$

where the form of $\tilde{R}_{ij}{}^k{}_l$ is fixed by maximal symmetry to

$$\tilde{R}_{lj}{}^k{}_i = k(\delta^k_l \tilde{g}_{ij} - \delta^k_j \tilde{g}_{il}).$$

The non-zero components of the Ricci tensor are then

$$R_{00} = -3\frac{a''}{a}, \qquad R_{ij} = \left(\frac{a''}{a} + 2\frac{a'^2}{a^2} + \frac{2k}{a^2}\right) g_{ij},$$

and the Ricci scalar is

$$R = \frac{6}{a^2}(aa'' + a'^2 + k).$$

Putting these together, the Einstein tensor has components

$$\begin{aligned} G_{00} &= 3\left(\frac{a'^2}{a^2} + \frac{k}{a^2}\right), \\ G_{ij} &= -\left(\frac{2a''}{a} + \frac{a'^2}{a^2} + \frac{k}{a^2}\right) g_{ij}, \\ G_{0i} &= 0. \end{aligned}$$

Using these in the Einstein equation $G_{ab} = 8\pi T_{ab}$ then gives

$$\begin{aligned} 00: \qquad & 3\left(\frac{a'^2}{a^2} + \frac{k}{a^2}\right) = 8\pi\rho, \\ ij: \qquad & -\left(\frac{2a''}{a} + \frac{a'^2}{a^2} + \frac{k}{a^2}\right) = 8\pi p. \end{aligned}$$

The first of these matches the first equation in the question. The second matches the second equation in the question on eliminating the $(a'/a)^2$ term using the first equation.

Multiply the first equation by a^2, differentiate with respect to τ and eliminate a'' using the second equation to show that

$$\rho' + 3\frac{a'}{a}(\rho + P) = 0.$$

Start with

$$\begin{aligned} a'^2 &= \frac{8\pi\rho}{3}a^2 - k \\ &\Downarrow \\ 2a'a'' &= \frac{8\pi}{3}\rho' a^2 + 2\frac{8\pi\rho}{3}aa' - 0 \\ &\Downarrow \\ 0 &= \rho' + 3\frac{a'}{a}(\rho + p). \end{aligned}$$

Derive the same equation directly from the local conservation of energy and momentum $\nabla^a T_{ab} = 0$.

We start from

$$\begin{aligned}
\nabla^a T_{ab} &= 0 \\
&= \nabla^a \left((p+\rho)u_a u_b + p g_{ab}\right) \\
&= \nabla^a (p+\rho) u_a u_b + (p+\rho)\nabla^a u_a u_b + (p+\rho) \underbrace{\cancel{u_a \nabla^a u_b}}_{\text{geodesic}} + \nabla^a p g_{ab} + p \underbrace{\cancel{\nabla^a g_{ab}}}_{\text{metric}} \\
&= u^a \nabla_a (p+\rho) u_b + (p+\rho)\nabla_a u^a u_b + \nabla_b p \\
&= u^a \partial_a (p+\rho) u_b + (p+\rho)(\partial_a u^a + \Gamma^a{}_{ac} u^c) u_b + \partial_b p.
\end{aligned}$$

Now set $b = 0$ and use $u^a = (1, 0, 0, 0)$ in a comoving coordinate frame (such as that in which the metric takes the form $\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \ldots$)

$$\begin{aligned}
\nabla^a T_{a0} &= u^0 \partial_0 (p+\rho) u_0 + (p+\rho)(\partial_0 u^0 + \Gamma^a{}_{a0} u^0) u_0 + \partial_0 p \\
&= -p' - \rho' - (p+\rho)\Gamma^a{}_{a0} + p' \\
&= -p' - \rho' - (p+\rho)\frac{a'}{a}\delta^i_i + p' \\
&= -\rho' - 3(p+\rho)\frac{a'}{a} \\
&\equiv 0,
\end{aligned}$$

which is the same equation.

7. Cosmological constant

A cosmological constant Λ modifies Einstein's equations as follows

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}.$$

Show that the cosmological constant is mathematically equivalent to a perfect fluid with density $\rho_\Lambda = \Lambda/8\pi$ and pressure $P_\Lambda = -\Lambda/8\pi$.

We have

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$
$$\Downarrow$$
$$G_{ab} = 8\pi \left(T_{ab} - \frac{\Lambda}{8\pi} g_{ab} \right),$$

so that

$$T^\Lambda_{ab} = -\frac{\Lambda}{8\pi} g_{ab}.$$

For a perfect fluid we have

$$T_{00} = \rho \equiv -\frac{\Lambda}{8\pi} g_{00} = \frac{\Lambda}{8\pi}.$$

As T^{Λ}_{ab} has only a g_{ab} term, we must have $\rho = -p$.

Hence show that for cosmological solutions we have

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}, \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}.$$

Pressure and density are extensive so add the contribution of a cosmological constant into the equations for a' and a'' derived in the last question to give

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}, \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P - 2\frac{\Lambda}{8\pi}),$$

which are the sought after equations.

In an expanding universe with contributions from pressureless matter, radiation and a cosmological constant, which contribution will dominate the energy density at a) early times; b) late times? Consider a universe with a positive cosmological constant: how does the scalar factor $a(\tau)$ behave at late times? Does this universe have a future horizon?

In an expanding universe where $a' > 0$ we saw in the lectures that the density of pressureless matter, radiation and a cosmological constant go as a^{-3}, a^{-4} and a^0. At early times (where a is small), the a^{-4} contribution will dominate – the universe will be radiation dominated. At late times (where a is large), the constant contribution will dominate – the universe will be dominated by the cosmological constant.

With $k = 0$ and a positive cosmological constant, $\Lambda > 0$, we have

$$\left(\frac{a'}{a}\right)^2 = \frac{\Lambda}{3}$$
$$a(\tau) \sim \mathrm{e}^{\sqrt{\Lambda/3}\tau}.$$

Define the conformal time as

$$\mathrm{d}\eta = a^{-1}\mathrm{d}\tau \sim \mathrm{e}^{-\sqrt{\Lambda/3}\tau}\mathrm{d}\tau.$$

A future horizon exists if η does not continue to $+\infty$ or equivalently if the following integral converges

$$\begin{aligned}\eta_\infty - \eta_{\tau'} &\sim \int_{\tau'}^{\infty} \mathrm{e}^{-\sqrt{\Lambda/3}\tau}\mathrm{d}\tau \\ &\sim -\frac{1}{\sqrt{\Lambda/3}}\mathrm{e}^{-\sqrt{\Lambda/3}\tau}|_{\tau'}^{\infty} \\ &\sim -\frac{1}{\sqrt{\Lambda/3}}(1 - \mathrm{e}^{-\sqrt{\Lambda/3}\tau'}).\end{aligned}$$

This does indeed converge to a finite value, so there is a future horizon.

8.* Conformal transformations

A Weyl transformation of a spacetime is one where the metric g_{ab} of the original spacetime is transformed into the metric $\tilde{g}_{ab}$ of a new spacetime, such that they are related by

$$\tilde{g}_{ab} = \Omega^2 g_{ab},$$

where Ω is a function of the spacetime coordinates x^a. Compute $\tilde{\Gamma}^a{}_{bc}$ and show that $\tilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc}$ if and only if Ω is constant.

We use $\tilde{g}^{ab} = \Omega^{-2} g^{ab}$ and the formula for the connection coefficients

$$\begin{aligned}
\tilde{\Gamma}^a{}_{bc} &= \tfrac{1}{2}\tilde{g}^{ad}(\partial_b \tilde{g}_{cd} + \partial_c \tilde{g}_{bd} - \partial_d \tilde{g}_{bc}) \\
&= \tfrac{1}{2}\Omega^{-2} g^{ad}(\Omega^2 \partial_b g_{cd} + \Omega^2 \partial_c g_{bd} - \Omega^2 \partial_d g_{bc} + g_{cd}\partial_b \Omega^2 + g_{bd}\partial_c \Omega^2 - g_{bc}\partial_d \Omega^2) \\
&= \tfrac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) + \Omega^{-1} g^{ad}(g_{cd}\partial_b \Omega + g_{bd}\partial_c \Omega - g_{bc}\partial_d \Omega) \\
&= \Gamma^a{}_{bc} + \Omega^{-1}(g^{ad} g_{cd}\partial_b \Omega + g^{ad} g_{bd}\partial_c \Omega - g^{ad} g_{bc}\partial_d \Omega) \\
&= \Gamma^a{}_{bc} + \Omega^{-1}(\delta^a_c \partial_b \Omega + \delta^a_b \partial_c \Omega - g_{bc}\partial^a \Omega).
\end{aligned}$$

Thus the coefficients agree if and only if Ω is constant.

Suppose in the original spacetime one has a solution to the source-free Maxwell equations

$$\nabla_a F^{ab} = 0, \qquad \nabla_{[a} F_{bc]} = 0.$$

Show that F_{ab} is also a solution to the source-free Maxwell equations in the new spacetime with metric $\tilde{g}_{ab}$ (with a corresponding connection $\tilde{\nabla}_a$) provided spacetime is four dimensional. You may wish to use $\Gamma^a{}_{ab} = (-g)^{-1/2}\partial_b (-g)^{1/2}$ where $g = \det g_{ab}$.

Note that F_{ab} is a solution in both spacetimes but the tensor with the indices raised is different for each – $\tilde{F}^{ab} = \tilde{g}^{ac}\tilde{g}^{bd} F_{bd}$ while $F^{ab} = g^{ac} g^{bd} F_{bd}$. First note that the expression for the trace of the Christoffel symbol allow us to write

$$\nabla_a T^{ab} = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}\, T^{ab}),$$

where T^{ab} is an arbitrary antisymmetric tensor. The first equation can the be rewritten as

$$\begin{aligned}
\tilde{\nabla}_a \tilde{F}^{ab} &= \frac{1}{\sqrt{-\tilde{g}}}\partial_a(\sqrt{-\tilde{g}}\tilde{F}^{ab}) \\
&= \frac{1}{\sqrt{-\tilde{g}}}\partial_a(\sqrt{-\tilde{g}}\tilde{g}^{ac}\tilde{g}^{bd}F_{cd}) \\
&= \frac{1}{\Omega^D\sqrt{-g}}\partial_a(\Omega^D\sqrt{-g}\Omega^{-4}g^{ac}g^{bd}F_{cd}) \\
&= \frac{1}{\Omega^D\sqrt{-g}}\partial_a(\Omega^{D-4}\sqrt{-g}F^{ab}) \\
&= \Omega^{-4}\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}F^{ab}) + F^{ab}\frac{1}{\Omega^D}\partial_a\Omega^{D-4},
\end{aligned}$$

from which we see F_{ab} solves the Maxwell equations in the new spacetime (given that it solves it in the old spacetime) provided $D = 4$.

Let $\tilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + A^a{}_{bc}$, where A is symmetric on the lower indices. The second equation is

$$\begin{aligned}
\tilde{\nabla}_{[a}F_{bc]} &= \nabla_{[a}F_{bc]} - A^d{}_{[ab}F_{|d|c]} - A^d{}_{[ac}F_{b]d} \\
&= \nabla_{[a}F_{bc]} + A^d{}_{[ab}F_{c]d} + A^d{}_{[ab}F_{c]d} \\
&= \nabla_{[a}F_{bc]} + 2A^d{}_{[ab}F_{c]d} \\
&= \nabla_{[a}F_{bc]},
\end{aligned}$$

where the final term drops out as A is symmetric on the lower indices.

The metric for a flat FRW spacetime ($k = 0$) is sometimes written as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{2/3}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2).$$

Show that this spacetime is conformal to Minkowski spacetime.

We rewrite the metric as

$$\mathrm{d}s^2 = \left(\frac{\tau}{\tau_0}\right)^{2/3}\left(-\left(\frac{\tau}{\tau_0}\right)^{-2/3}\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right).$$

Defining

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{1/3}, \qquad \mathrm{d}t = a^{-1}\mathrm{d}\tau,$$

we have

$$\mathrm{d}s^2 = a(\tau)^2\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right).$$

We see this metric is conformal to Minkowski spacetime with a conformal factor $\Omega(t)^2 = a(\tau)^2$.

This conformal factor is defined by

$$\mathrm{d}t = a^{-1}\mathrm{d}\tau = \left(\frac{\tau}{\tau_0}\right)^{-1/3}\mathrm{d}\tau.$$

Integrating both sides we have

$$t = \tfrac{3}{2}\tau^{2/3}\tau_0^{1/3} + c_0,$$

where the constant of integration can be set to zero by shifting the origin of t. We then have

$$a(\tau)^2 = \left(\frac{\tau}{\tau_0}\right)^{2/3} = \tfrac{2}{3}t\tau_0^{-1} = \Omega(t)^2.$$

Using this, or otherwise, find a formula for the cosmological red shift of light emitted by a galaxy at $\tau = \tau_1$ and measured by an observer at $\tau = \tau_2$. Assume both the source and observer are comoving.

Light travels on null geodesics with $\mathrm{d}s^2 = 0$. In the conformal coordinates we have

$$\Omega^2(-\mathrm{d}t^2 + \mathrm{d}r^2) = 0,$$

where $\mathrm{d}r^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, so that $\mathrm{d}t = \mathrm{d}r$ for light. In other words

$$r = t + \text{const}.$$

Let the light source be at the origin and our observer be at a position r, both at rest with respect to the comoving frame with time τ. Using the same argument as in the lectures, one finds that the time interval between two consecutive waves is

$$\mathrm{d}\tau_{\text{source}} = a(\tau_{\text{source}})\mathrm{d}t,$$
$$\mathrm{d}\tau_{\text{obser}} = a(\tau_{\text{obser}})\mathrm{d}t,$$

so that

$$\frac{\lambda_{\text{obser}}}{\lambda_{\text{source}}} = \frac{\mathrm{d}\tau_{\text{source}}}{\mathrm{d}\tau_{\text{obser}}} = \frac{a(\tau_{\text{source}})}{a(\tau_{\text{obser}})} = \left(\frac{\tau_{\text{source}}}{\tau_{\text{obser}}}\right)^{1/3}.$$

The red shift is then

$$z = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\text{obser}} - \lambda_{\text{source}}}{\lambda_{\text{source}}} = \left(\frac{\tau_{\text{source}}}{\tau_{\text{obser}}}\right)^{1/3} - 1.$$

A slicker way to get to this answer is to note that as a Killing vector has a corresponding conserved quantity on geodesics, a *conformal* Killing vector corresponds to a conserved quantity on *null* geodesics. A conformal Killing vector is one that satisfies

$$\nabla_{(a}K_{b)} = f(x)g_{ab},$$

for some function $f(x)$. For the metric at hand, one can show that $K = \partial_t = a(\tau)\partial_\tau$ is a conformal Killing vector, and so we have a quantity $K_a\dot{x}^a$ that is conserved on null geodesics. This gives

$$K^a g_{ab}\dot{x}^b = -a(\tau)\omega(\tau) = \text{const},$$

where $\omega(\tau)$ is the frequency of the photon traveling on the null geodesic.

9.* Energy conditions

The *weak energy condition* (WEC) requires that any stress-energy tensor must satisfy

$$T_{ab}t^a t^b \geq 0,$$

for all timelike vectors t^a. Show that for a perfect fluid the WEC implies

$$\rho \geq 0, \qquad \rho + P \geq 0.$$

What type of matter would violate the WEC?

The stress tensor for a perfect fluid is

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab}.$$

Consider $T_{ab}t^a t^b$ for some timelike vector t^a. In the local inertial frame comoving with the fluid (in which $u^a = (1,0,0,0)$ and $g_{ab} = \eta_{ab}$) we pick

$$t^a = \gamma(v)(1, v, 0, 0).$$

Expanding out we find

$$\begin{aligned} T_{ab}t^a t^b &= (p+\rho)\gamma^2 - p \\ &= \gamma^2(\rho + v^2 p). \end{aligned}$$

This must be greater than or equal to zero for all $0 \leq v < 1$. For $v = 0$ this implies $\rho \geq 0$ and as $v \to 1$ we must have $\rho + p \geq 0$.

The *strong energy condition* (SEC) requires that any stress-energy tensor must satisfy

$$T_{ab}t^a t^b \geq \tfrac{1}{2} T^a{}_a t^b t_b,$$

for all timelike vectors t^a. Show that for a perfect fluid the SEC implies

$$\rho + P \geq 0, \qquad \rho + 3P \geq 0.$$

In the local inertial frame we have

$$\begin{aligned} T^a{}_a &= (p+\rho)u_a u_b \eta^{ab} + p\eta_{ab}\eta^{ab} \\ &= -(p+\rho) + 4p \\ &= 3p - \rho. \end{aligned}$$

The SEC is then

$$\begin{aligned} 0 &\le T_{ab}t^a t^b - \tfrac{1}{2}T^a{}_a t^b t_b \\ &\le (p+\rho)\gamma^2 - p - \tfrac{1}{2}(3p-\rho)(-1) \\ &\le (p+\rho)\gamma^2 + \tfrac{1}{2}(p-\rho) \\ &\le \gamma^2\left((\rho+3p) + v^2(\rho-p)\right). \end{aligned}$$

As this must hold for $0 \le v < 1$, we get

$$\rho + 3p \ge 0, \qquad \rho + p \ge 0.$$

Does the SEC imply the WEC? Is the SEC satisfied for a flat FRW universe with a positive cosmological constant?

The SEC does not imply the WEC. Small negative ρ with large positive p obey the SEC but violate the WEC. Conversely, positive ρ with negative p where $3|p| > \rho$ obey the WEC but violate the SEC.

A flat FRW universe with a positive cosmological constant Λ has $\rho_\Lambda = \Lambda/8\pi$ and pressure $p_\Lambda = -\Lambda/8\pi$. This does not obey the second condition of the SEC.

Starting from the Einstein equations, show that the SEC implies

$$R_{ab}t^a t^b \ge 0,$$

where R_{ab} is the Ricci tensor.

The Einstein equations are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = 8\pi T_{ab}.$$

Contracting with g_{ab} gives

$$R - 4\tfrac{1}{2}R = -R = 8\pi T^a{}_a,$$

allowing us to write

$$R_{ab} = 8\pi(T_{ab} - \tfrac{1}{2}T^a{}_a g_{ab}).$$

Contracting with an arbitrary timelike vector t^a, the right-hand side of this expression is greater than or equal to zero due to the SEC. This implies

$$R_{ab}t^a t^b \geq 0.$$

10.* Cosmic strings

Consider a static, infinitely long, cylindrically symmetric matter distribution of constant radius that is invariant under Lorentz boosts along the symmetry axis. Using similar arguments that lead to the Schwarzschild solution, show that the metric outside the matter distribution can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (\alpha + \beta r)^2 \mathrm{d}\phi^2 + \mathrm{d}z^2,$$

where r is the radial direction, z is the direction along the symmetry axis, and α and β are constants.

This follows the derivation of the Schwarzschild solution closely. The idea is to start with an ansatz for the the metric that incorporates rotational symmetry in the ϕ direction, translation symmetry in the z direction, and no time dependence so that we have

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + C(r)\mathrm{d}\phi^2 + D(r)\mathrm{d}z^2.$$

Off-diagonal components in $\mathrm{d}t\mathrm{d}r$ and $\mathrm{d}z\mathrm{d}r$ can be eliminated by defining the z and t coordinates. One then follows the Schwarzschild derivation to fix the unknown functions using symmetry or coordinate freedom.

For the case $\alpha = 0$, consider the spacelike surfaces defined by $t =$ const.and $z =$ const. Calculate the circumference of a circle of constant coordinate radius r in such a surface.

For $\mathrm{d}t = \mathrm{d}z = 0$ we have

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \beta^2 r^2 \mathrm{d}\phi^2.$$

A circle of constant radius has $\mathrm{d}r = 0$ so

$$\mathrm{d}s^2 = \mathrm{d}l^2 = \beta^2 r^2 \mathrm{d}\phi^2,$$

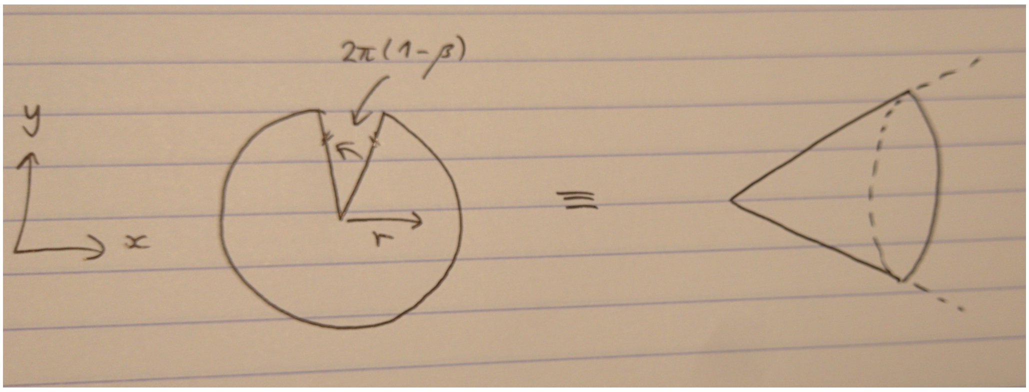
where l is the proper distance of the circle. For constant r we can integrate directly to get

$$l = 2\pi\beta r.$$

The circumference of the circle is $2\pi\beta r$, that is the usual $2\pi r$ multiplied by β.

Argue that the for $\beta < 1$, the geometry on the spacelike surface is that of a two-dimensional cone embedded in three-dimensional Euclidean space.

For $\beta < 1$ there is a deficit angle – we can define $\phi' = \beta\phi$ and let $\phi' \in (0, 2\pi\beta)$. If we let $r^2 = x^2 + y^2$ this metric describes locally flat Euclidean space with a wedge of angular size $2\pi(1 - \beta)$ removed and the edges of the wedge identified. Joining the edges of the wedge together, this equivalently describes the geometry of a 2d cone embedded into 3d Euclidean space.



10.* Totally antisymmetric torsion

Let ∇_a be a torsion-free connection and define a new connection $\tilde{\nabla}_a$ such that

$$\tilde{\nabla}_a V^b = \nabla_a V^b - \tfrac{1}{2} T^b{}_{ac} V^c,$$

for any vector field V^a, where $T^b{}_{ac} = -T^b{}_{ca}$. Let $\Omega_{ab} = -\Omega_{ba}$ be a non-degenerate antisymmetric tensor with inverse $\hat{\Omega}^{ac}$ such that

$$\Omega_{ab} \hat{\Omega}^{bc} = \delta^c_a.$$

Show that there exists a unique choice for $T^b{}_{ac}$ such that

$$\tilde{\nabla}_a \Omega_{bc} = 0.$$

We begin with noting that the definition of the shifted connection means we have

$$\tilde{\nabla}_a \Omega_{bc} = \nabla_a \Omega_{bc} + \tfrac{1}{2} T^d{}_{ab} \Omega_{dc} + \tfrac{1}{2} T^d{}_{ac} \Omega_{bd}. \tag{3}$$

This can be derived by checking how the shifted connection acts on scalars and one-forms, then using induction.

We want to isolate a single $T^a{}_{bc}$ so that we can solve for it. To do this, we play the same game we played when solving for the Levi-Civita connection. Let (3) be (a), with (b) and (c) the same expressions with $(abc) \to (bca)$ and $(abc) \to (cab)$. Taking $(a) + (b) - (c)$ we find

$$
\begin{aligned}
0 \equiv \tilde{\nabla}_a \Omega_{bc} + \tilde{\nabla}_b \Omega_{ca} - \tilde{\nabla}_c \Omega_{ab} &= \nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab} \\
&\quad + \tfrac{1}{2} T^d{}_{ab} \Omega_{dc} + \tfrac{1}{2} T^d{}_{ac} \Omega_{bd} \\
&\quad + \tfrac{1}{2} T^d{}_{bc} \Omega_{da} + \tfrac{1}{2} T^d{}_{ba} \Omega_{cd} \\
&\quad - \tfrac{1}{2} T^d{}_{ca} \Omega_{db} - \tfrac{1}{2} T^d{}_{cb} \Omega_{ad} \\
&= \nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab} \\
&\quad + \tfrac{1}{2} T^d{}_{ab} \Omega_{dc} + \tfrac{1}{2} T^d{}_{ac} \Omega_{bd} \\
&\quad + \tfrac{1}{2} T^d{}_{bc} \Omega_{da} + \tfrac{1}{2} T^d{}_{ab} \Omega_{dc} \\
&\quad - \tfrac{1}{2} T^d{}_{ac} \Omega_{bd} - \tfrac{1}{2} T^d{}_{bc} \Omega_{da} \\
&= \nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab} + T^d{}_{ab} \Omega_{dc}.
\end{aligned}
$$

Multiplying through by $\hat{\Omega}^{ce}$ we get

$$
\begin{aligned}
0 &= (\nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab}) \hat{\Omega}^{ce} + T^d{}_{ab} \Omega_{dc} \hat{\Omega}^{ce} \\
&= (\nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab}) \hat{\Omega}^{ce} + T^e{}_{ab}.
\end{aligned}
$$

This uniquely fixes $T^e{}_{ab}$ in terms of ∇ and Ω. Now one needs to check that this choice sets $\tilde{\nabla}_a \Omega_{bc} = 0$ and not just the particular combination that appears above. We have

$$
\begin{aligned}
\tilde{\nabla}_a \Omega_{bc} &= \nabla_a \Omega_{bc} - \tfrac{1}{2} (\nabla_a \Omega_{be} + \nabla_b \Omega_{ea} - \nabla_e \Omega_{ab}) \hat{\Omega}^{ed} \Omega_{dc} - \tfrac{1}{2} (\nabla_a \Omega_{ce} + \nabla_c \Omega_{ea} - \nabla_e \Omega_{ac}) \hat{\Omega}^{ed} \Omega_{bd} \\
&= \nabla_a \Omega_{bc} - \tfrac{1}{2} (\nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab}) + \tfrac{1}{2} (\nabla_a \Omega_{cb} + \nabla_c \Omega_{ba} - \nabla_b \Omega_{ac}) \\
&= \nabla_a \Omega_{bc} - \tfrac{1}{2} (\nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} - \nabla_c \Omega_{ab}) + \tfrac{1}{2} (-\nabla_a \Omega_{bc} - \nabla_c \Omega_{ab} + \nabla_b \Omega_{ca}) \\
&= \nabla_a \Omega_{bc} - \nabla_a \Omega_{bc} \\
&= 0,
\end{aligned}
$$

so it is indeed the unique choice for $T^e{}_{ab}$ that sets $\tilde{\nabla}_a \Omega_{bc} = 0$.