16.1 - Cosmological metric:

Recall

$$ds^2 = -d\tau^2 + a(\tau)^2 \begin{cases} d\psi^2 + \sin^2\psi \, d\Omega^2 \\ d\psi^2 + \psi^2 \, d\Omega^2 \\ d\psi^2 + \sinh^2\psi \, d\Omega^2 \end{cases}$$

Scale factor $a(\tau)$ obeys

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}$$

$$\frac{a''}{a} = -\frac{4\pi}{3}(p + 3\rho)$$

$$\rho' + 3(\rho + p)\frac{a'}{a} = 0$$

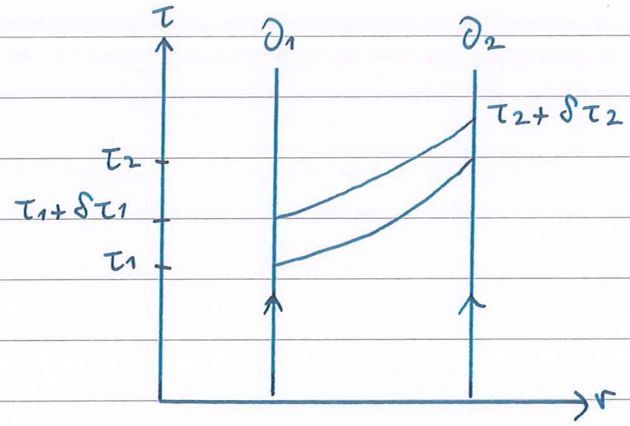
$$k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

Setting $r = \begin{cases} \sin\psi \\ \psi \\ \sinh\psi \end{cases}$ gives

$$ds^2 = -d\tau^2 + a^2(\tau)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right)$$

16.2 - Gravitational redshift

Two co-moving observers O_1 and O_2



Photon travels on null geodesic

$$\mathcal{L} = -\dot{\tau}^2 + a^2 \frac{\dot{r}^2}{1-kr^2} = 0$$

$$\Rightarrow \quad \frac{\dot{\tau}}{a} = \pm \frac{\dot{r}}{(1-kr^2)^{1/2}}$$

Take + for $r_2 > r_2$ geodesic.

$$\Rightarrow \int_{\tau_1}^{\tau_2} \frac{d\tau}{a(\tau)} = \int_{r_1}^{r_2} \frac{dr}{(1-kr^2)^{1/2}}$$

RHS depends on (r_2, r_1) only

$$\Rightarrow \int_{\tau_1}^{\tau_2} \frac{d\tau}{a(\tau)} = \int_{\tau_1+\delta\tau_1}^{\tau_2+\delta\tau_2} \frac{d\tau}{a(\tau)}$$

$$\Rightarrow \int_{\tau_1}^{\tau_1+\delta\tau_1} \frac{d\tau}{a(\tau)} = \int_{\tau_2}^{\tau_2+\delta\tau_2} \frac{d\tau}{a(\tau)}$$

As $\quad d\tau_1 , \delta\tau_2 \to 0$

$$\frac{\delta\tau_1}{\tau_1} = \frac{\delta\tau_2}{\tau_2}$$

O_1 emits photon with $\delta\tau_1$ between peaks,
O_2 measures it with $\delta\tau_2$ between peaks

$$E_i = \frac{2\pi\hbar}{\delta t_i}$$

$$\Rightarrow \frac{E_2}{E_1} = \frac{a(\tau_1)}{a(\tau_2)}$$

"Gravitational redshift in expanding universe"

$a(\tau_2) > a(\tau_1)$ for expansion $\Rightarrow$ $E_2 < E_1$

<u>NB</u> : Cosmologists write $\dfrac{E_2}{E_1} = \dfrac{1}{1+z}$

z known as the <u>redshift</u>.

16.3 - Hubble's law

Consider photon emitted from galaxy nearly at $\tau_1 = \tau_0 - \Delta\tau$ and received at $\tau_2 = \tau_0$

$$a(\tau_0 - \Delta\tau) \simeq a(\tau_0) - \Delta\tau\, a'(\tau_0) + \mathcal{O}(\Delta\tau^2)$$

But
$$z = \frac{a(\tau_2)}{a(\tau_1)} - 1$$

$$= \frac{a(\tau_0)}{a(\tau_0 - \Delta\tau)} - 1$$

$$= \Delta\tau \frac{a'(\tau_0)}{a(\tau_0)} + \mathcal{O}(\Delta\tau^2)$$

$$\simeq \Delta\tau\, H_0$$

$H_0 = \dfrac{a'(\tau_0)}{a(\tau_0)}$ is <u>Hubble's constant</u>

For $k = 0$, <u>physical distance</u> travelled by photon is

$$d = \int_{r_1}^{r_2} a(\tau_0)\, dr$$

"present instaneous proper distance"

$$\simeq a(\tau_0) \int_{\tau_0 - \Delta\tau}^{\tau_0} \frac{d\tau}{a(\tau)}$$

$$\approx \Delta\tau + \mathcal{O}(\Delta\tau^2)$$

$$\Rightarrow \quad z = H_0 d$$

"Hubble's law"

As $H_0 \sim$ constant, amount of redshift gives measure of distance to emitting objects

16.4 - Horizons

How much of the universe can be observed by a single co-moving observer?

Focus on $k=0$. Study null geodesics.

Define conformal time

$$d\eta = \frac{d\tau}{a}$$

$$\eta = \int^{\tau} \frac{d\tau'}{a(\tau')}$$

Metric is then

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dx^i dx^i\right)$$

Flat metric on subset of $\mathbb{R}^4$ (might be horizons!) given by (η_{ab}, U), $U \subset \mathbb{R}^4$

This metric is conformally flat, light rays travel at 45° on spacetime diagrams.

The <u>causal structure</u> (how light behaves) is then easy to analyse.

Since $a^2 > 0$, events connected by a null geodesic iff connected by a geodesic in flat spacetime.

Need to worry about the range of η to see how $U \subset \mathbb{R}^4$.

If $\displaystyle\int^{\infty} \frac{d\tau}{a} \to \infty$ and $\displaystyle\int_{-\infty} \frac{d\tau}{a} \to -\infty$

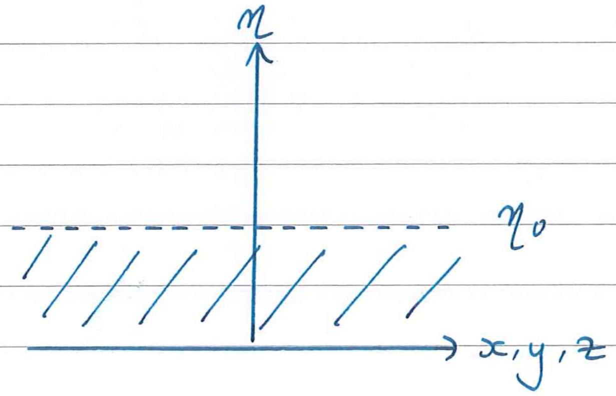
then $\eta \in (-\infty, \infty)$ so $U = \mathbb{R}^4$

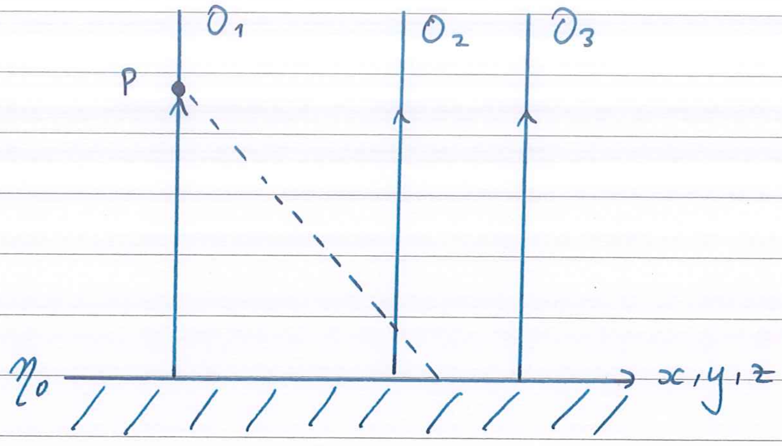
Otherwise there are <u>horizons</u>.

- If $\displaystyle\int_0 \frac{d\tau}{a}$ converges / is finite, there is a <u>past horizon</u> ~~(a grows as $\tau \to 0$, universe contracts going forward)~~

e.g. $\displaystyle\int_0^{\tau} \frac{d\tau}{a} = \eta(\tau) - \eta_0$

a "particle horizon"





O_1 at P can receive signal from O_2 but not O_3.

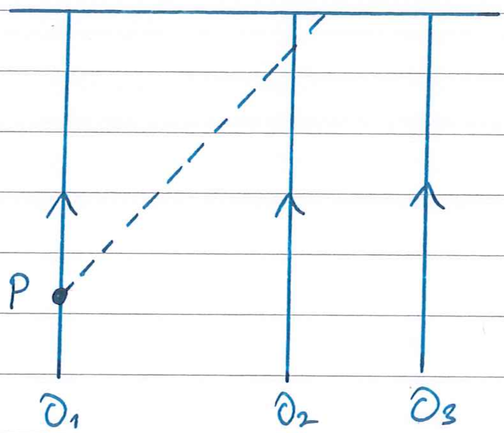
<u>Past horizon distance</u>:

$$a(\tau_0)\int_0^{\tau_0} \frac{d\tau}{a(\tau)}$$ is maximum physical distance at time τ_0 between observers in causal contact.

"distance travelled by light between τ_0 and start of universe"

- If $\int^{\infty} \frac{d\tau}{a}$ converges, there is a

<u>future horizon</u> ($a \to 0$ ~~as $\tau \to \infty$~~



Observer at P can send signal to O_2 but can never communicate with O_3

<u>Future horizon distance</u>:

$$a(\tau_0) \int_{\tau_0}^{\infty} \frac{d\tau}{a}$$

is maximum physical distance at time τ_0 between observers that can communicate in the future.

Existence of the horizons depends on the form of $a(\tau)$ (so the matter content and $k = \pm 1, 0$).

<u>Example</u>: Dust $(p = 0)$

$$a(\tau) = \alpha \tau^{2/3}$$

- $$\int_0^{\tau_0} \frac{d\tau}{a} \sim \alpha \tau^{1/3} \Big|_0^{\tau_0}$$
$$\sim \alpha \tau_0^{1/3}$$

Converges so there is a past horizon

- $$\int_{\tau_0}^{\infty} \frac{d\tau}{a} \sim \alpha \tau^{1/3} \Big|_{\tau_0}^{\infty} \sim \infty - \alpha \tau_0^{1/3}$$

Diverges, no future horizon.

- Past horizon distance

$$a(\tau_0)\int_0^{\tau_0}\frac{d\tau}{a(\tau)} = \alpha\tau_0^{2/3}\int_0^{\tau_0}\frac{1}{\alpha}\,\tau^{-2/3}\,d\tau$$

$$= 3\tau_0$$

$$(= 3c\tau_0)$$

Defines the size of the observable universe.

Example: Cosmological constant, $\Lambda > 0$

$$a(\tau) = e^{\alpha\tau} \qquad \alpha := \sqrt{\frac{\Lambda}{3}}$$

Both integrals converge so future and past horizons

Past: $$a(\tau_0)\int_0^{\tau_0}\frac{d\tau}{a(\tau)} = \frac{1}{\alpha}\left(e^{\alpha\tau_0}-1\right)$$

$$\simeq \frac{1}{\alpha}e^{\alpha\tau_0} \qquad \left(\tau_0 \gg \frac{1}{\alpha}\right)$$

Grows exponentially with τ_0 (can observe more of universe)

Future: $$a(\tau_0)\int_{\tau_0}^{\infty}\frac{d\tau}{a(\tau)} = \frac{1}{\alpha}e^{\alpha\tau_0}\left(e^{-\alpha\tau_0}-0\right)$$

$$= \frac{1}{\alpha}$$

Constant! Cannot communicate with different distances as τ_0 changes