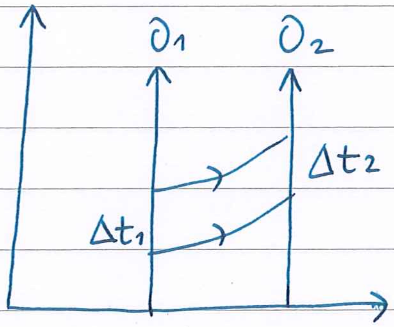
12.1 - Experimental Tests

- Gravitational redshift
- Perihelion shift
- Bending of light

Gravitational Redshift

(See lecture 10)



$$\Delta t_1 = \Delta t_2$$

$$\frac{\Delta \tau_1}{\Delta \tau_2} = \left(\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}} \right)^{1/2}$$

If curves represent the crests of an EM wave, the energies measured by O_1 and O_2 are

$$E_1 = \frac{2\pi\hbar}{\Delta \tau_1} \quad , \quad E_2 = \frac{2\pi\hbar}{\Delta \tau_2}$$

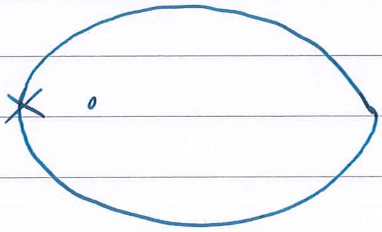
$$\frac{E_2}{E_1} = \left(\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}} \right)^{1/2}$$

If $r_2 > r_1$, $\frac{E_2}{E_1} < 1$

Photons redshifted as they climb out of a gravitational potential.

12.3 – Perihelion Shift

Perihelion is point of closest approach of orbit around sun



Recall that for TL geodesic in Schwarzschild

Observation: perihelion shifts with each orbit. Newton predicts 0 of the shift observed (43 arcseconds / century)

$$V(r) = -\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}, \quad \frac{E^2-1}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

Consider a circular orbit at $r = r_+$ with small perturbations

$$0 > \frac{E^2 - 1}{2} > V(r_+) \quad , \quad$$ ~~E^2-1~~

Step 1 :

Circular $\Rightarrow \quad \ddot{r} = 0 = \frac{\partial V}{\partial r}$

$$\Rightarrow \quad J^2 = \frac{Mr^2}{r - 3M}$$

For $r \gg M$, $\quad J^2 \simeq Mr$

Step 2 : small perturbations around orbit.

Convenient to change $(r, \tau) \mapsto (u, \phi)$

$$u \equiv \frac{M}{r}$$

$$\frac{du}{d\phi} = -\frac{M}{r^2}\frac{dr}{d\phi}$$

$$= -\frac{M}{r^2}\frac{\dot{r}}{\dot{\phi}}$$

$$= -\frac{M\dot{r}}{J}$$

Rewrite equation relating conserved quantities

$$\frac{E^2-1}{2} = \frac{1}{2}\left(\frac{J}{M}\frac{du}{d\phi}\right)^2 - u + \frac{J^2}{2M^2}u^2 - \frac{J^2u^3}{M^2}$$

Take 2nd derivative

$$\frac{d^2u}{d\phi^2} = \frac{du}{d\phi}\frac{d}{du}\left(\frac{du}{d\phi}\right)$$

$$= \frac{1}{2}\frac{d}{du}\left(\frac{du}{d\phi}\right)^2$$

$$= \underbrace{\frac{M^2}{J^2} - u}_{\text{Newton}} + \underbrace{3u^2}_{\text{GR}}$$

<u>Perturb</u>: $u(\phi) = \frac{M}{R} + v(\phi)$

$\uparrow$ radius of circular orbit

$$\Rightarrow \frac{d^2v}{d\phi^2} = \frac{M^2}{J^2} - \left(\frac{M}{R}+v\right) + 3\left(\frac{M}{R}+v\right)^3$$

$$= \frac{M^2}{J^2} - \frac{M}{R} + 3\left(\frac{M}{R}\right)^2$$

$$- \left(1 - \frac{6M}{R}\right)v + O(v^2) .$$

From Step 1: $J^2 = \frac{Mr^2}{r-3M}$ the $O(v^0)$ term vanishes

So $$\frac{d^2v}{d\phi^2} + \left(1 - \frac{6M}{R}\right) v = 0$$

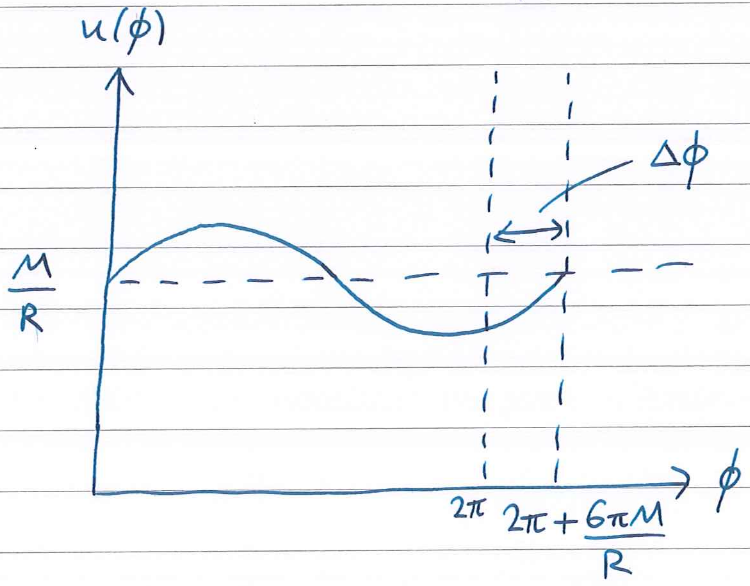
- $\underline{R < 6M}$: unstable (hyperbolic solⁿ)
- $\underline{R > 6M}$: stable (trig solⁿ)

For $R > 6M$, solution is periodic with frequency

$$\omega^2 = 1 - \frac{6M}{R}$$

$$T_\phi = \frac{2\pi}{\omega} = \frac{2\pi}{\left(1 - \frac{6M}{R}\right)^{1/2}}$$

$$\simeq 2\pi + \frac{6\pi M}{R} \quad \text{for } R \gg 3M$$



Point of closest approach shifts by $\frac{6\pi M}{R}$ on each orbit!

Restoring units : $\Delta\phi = \frac{6\pi G M}{c^2 R}$

For Mercury : $R = 5.55 \times 10^7$ km

M = mass of sun = 1.99×10^{30} kg

$\Delta\phi \approx 5 \times 10^{-7}$ radians per orbit.

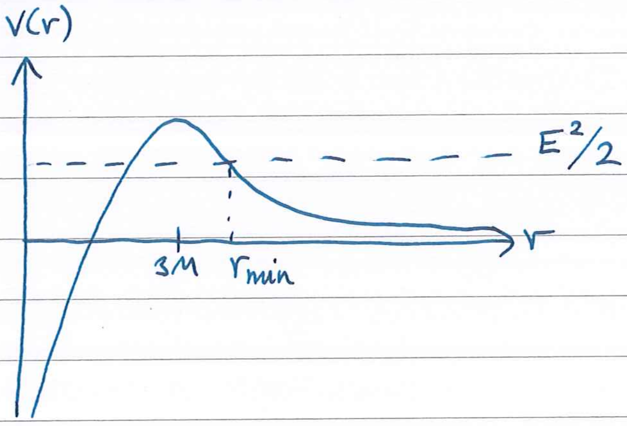
Matches observation!

12.4 - Bending of light.

$$\mathcal{L} = -K = 0$$

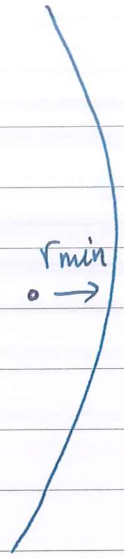
$$\Rightarrow \frac{E^2}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \frac{J^2}{2r^2}\left(1 - \frac{2M}{r}\right)$$



In region $0 < \frac{E^2}{2} < V(3M)$

$$E > \frac{J}{3M}$$



Again : $u = \frac{M}{r}$

$$\Rightarrow \frac{d^2u}{d\phi^2} = \underbrace{-u}_{\text{Newtonian}} + \underbrace{3u^2}_{\text{GR}}$$

For $r \gg M$, $u \ll 1$ and to leading order

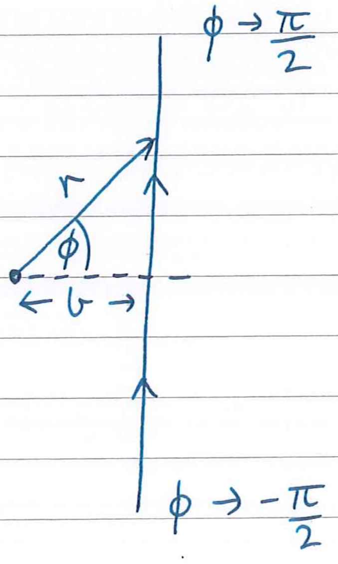
$$\frac{d^2u}{d\phi^2} = -u$$

$$\Rightarrow u(\phi) = A\cos(\phi - \phi_0)$$

Choose $\phi_0 = 0$

Substitute into $\left(\frac{ME}{J}\right)^2 = \left(\frac{du}{d\phi}\right)^2 + u^2$

$$\Rightarrow A = \frac{E}{J} M$$



"<u>Impact parameter</u>"

$$b = r\cos\phi$$

$$= \frac{M}{u}\cos\phi$$

$$= \frac{J}{E}$$

No deflection!

Now look at perturbation and include $\partial(u^2)$ term.

$$u(\phi) = A\cos\phi + v(\phi)$$

with limit : $A \sim \partial(\epsilon)$

$$v \sim \partial(\epsilon^2)$$

$$\epsilon \to 0$$

A large impact parameter and a small perturbation. $v << A$.

Look at eq$^{\underline{n}}$ for $\left(\frac{du}{d\phi}\right)^2$

$$0 = \left(\frac{du}{d\phi}\right)^2 + u^2 - 2u^3 - A^2$$

$$= \left(-A\sin\phi + v'\right)^2 + \left(A\cos\phi + v\right)^2$$

$$- 2\left(A\cos\phi + v\right)^3 - A^2$$

- Leading $\partial(\epsilon^2)$ term vanishes by construction
- Subleading terms give

$$0 = -2A\sin\phi\, v' + 2A\cos\phi\, v$$
$$- 2A^3\cos^3\phi$$

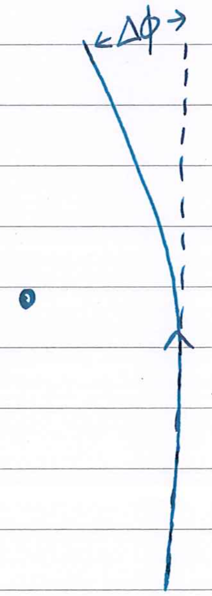
$$\Rightarrow \quad \sin\phi\, v' = \cos\phi\, v - A^2\cos^3\phi$$

$$\Rightarrow \quad v(\phi) = A^2(1+\sin^2\phi) + C\sin\phi$$

Choose $v(-\frac{\pi}{2}) = 0$ (no perturbation at ∞)

$$= 2A^2 - C$$

$$\Rightarrow v(\phi) = A^2(1+\sin\phi)^2$$



Deflection angle $\Delta\phi$ is $r \to \infty$ for $\phi > 0$

$$0 = u(\tfrac{\pi}{2} + \Delta\phi)$$

$$= 4A^2 - A\Delta\phi + \mathcal{O}(\Delta\phi^2)$$

$$\Rightarrow \Delta\phi = 4A$$

$$= \frac{4ME}{J}$$

$$= \frac{4M}{b}$$

Restoring units : $\Delta\phi = \frac{4GM}{bc^2}$

- Deflection larger for greater mass and closer approach.