Suggests that we can promate equations from SR to GR using

$$\partial_a \to D_a \qquad \text{"minimal coupling"}$$

which then reduce to SR laws in a local inertial frame.

Example: $D_a F^{ab} = 4\pi J^b$

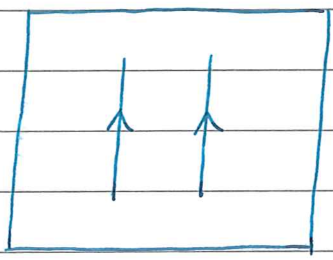
$D_{[a} F_{bc]} = 0$ Electromagnetism

$D_a T^{ab} = 0$ Stress tensor

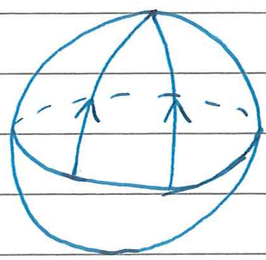
7.3 - Curvature

Deviations from special relativity at $\mathcal{O}(x^2)$ in g_{ab}

Characterised by focussing (or parting) of geodesics



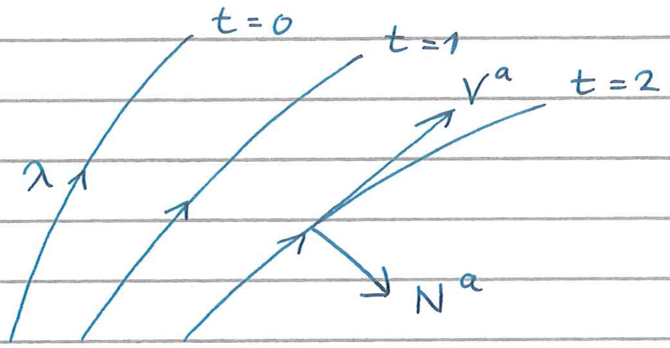
$\mathbb{R}^2$



S^2

Consider a 1-parameter family of timelike geodesics

$$x^a_t(\lambda) = x^a(\lambda, t)$$



$$V^a = \frac{dx^a}{d\lambda} \quad , \quad N^a = \frac{dx^a}{dt}$$

"Geodesics" $\Rightarrow$ $V^a D_a V^b = 0$ for all t.

We can change the affine parameter

$$\lambda \mapsto a(t)\lambda + b(t)$$

$$\Rightarrow \quad V^a \mapsto \frac{1}{a(t)} V^a$$

$$N^a \mapsto N^a + \left(a'(t)\lambda + b'(t) \right) V^a$$

Fix this using: 1) Norm of V^a preserved with t

2) Set $g(N, V) = 0$ (ignore deviation along geodesic)

1) $\frac{d}{d\lambda} g(V,V) = V^a \partial_a g(V,V)$

scalar

$= V^a D_a g(V,V)$

"metric" D_a

$= 2 V^a g_{bc} D_a V^b V^c$

$= 2 V_b \underbrace{V^a D_a V^b}_{0}$

$= 0$

$\Rightarrow \quad g(V,V) = -f(t)^2$ for some $f(t)$

Choose $a(t) = f(t)$ to fix

$g(V,V) = -1$

so that λ is proper time

2) $\frac{d}{d\lambda} g(V,N) = V^c D_c (g_{ab} V^a N^b)$

$Dg = 0$

$VDV = 0$

$= g_{ab} V^a V^c D_c N^b$

but $V^c D_c N^b - N^c D_c N^b = V^c \partial_c N^b - N^c \partial_c V^b$

need this

$= \frac{\partial}{\partial\lambda}\frac{dx^a}{dt} - \frac{\partial}{\partial t}\frac{\partial x^a}{\partial\lambda}$

$= 0$

So $\quad \dfrac{d}{d\lambda} g(V,N) = g_{ab} V^a N^c D_c V^b$

$$= \frac{1}{2} N^c D_c \big(\underbrace{g_{ab} V^a V^b}_{=-1 \ \forall \lambda, t} \big)$$

$$= 0$$

So $g(V,N)$ constant with λ.

Use remaining $\lambda \mapsto \lambda + b(t)$

$$N^a \mapsto N^a + b'(t) V^a$$

to fix $g(V,N) = 0$.

<u>Relative acceleration</u> of nearby geodesics

$$\frac{D^2 N^a}{D\lambda^2} = D_V D_V N^a$$

$$= V^b D_b (V^c D_c N^a)$$

$\downarrow \ [V,N]^a = 0$

$$= V^b D_b (N^c D_c V^a)$$

$$= V^b (D_b N^c)(D_c V^a) + V^b N^c D_b D_c V^a$$

$$= V^b (D_b N^c)(D_c V^a)$$

$$+ V^b N^c (D_b D_c - D_c D_b) V^a$$

$$+ V^b N^c D_c D_b V^a$$

$[N, V] = 0$

$$= N^b D_b V^c D_c V^a + V^b N^c D_c D_b V^a$$

$$+ V^b N^c [D_b, D_c] V^a$$

$= 0$ as geodesic.

$$= N^b D_b (V^c D_c V^a)$$

$$+ V^b N^c [D_b, D_c] V^a$$

$$\equiv + R_{bc}{}^a{}_d V^b N^c V^d$$

This defines the <u>Riemann curvature tensor</u>

$$+ R_{bc}{}^a{}_d V^d = [D_b, D_c] V^a$$

Clearly <u>C^∞ linear</u> in bc indices

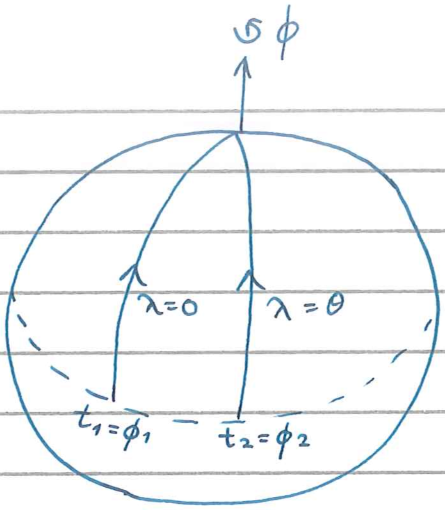
- Can check it is linear under

$V^a \to f V^a$ (c.f. $[D_a, D_b] f = 0$ for torsion free)

so transforms as a tensor.

<u>Example</u>: Round S^2

Identify $t = \phi$, $\lambda = \theta$



$$V^a = (V^\theta, V^\phi) = (1, 0)$$

$$N^a = (N^\theta, N^\phi) = (0, 1)$$

<u>Recall</u> : $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{DN^a}{D\lambda} = \frac{dx^b}{d\lambda} \nabla_b N^a$$

$$= V^b \nabla_b N^a$$

$$= \underbrace{\partial_\theta N^a}_{0} + \Gamma^a{}_{\theta\phi} N^\phi$$

$$\Rightarrow \quad \frac{DN^\phi}{D\lambda} = \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

Now $\quad \dfrac{D^2 N^\phi}{D\lambda^2} = \partial_\theta \left(\dfrac{DN^\phi}{D\lambda} \right) + \Gamma^\phi{}_{\theta\phi} \dfrac{DN^\phi}{D\lambda}$

$$= \partial_\theta \left(\frac{\cos\theta}{\sin\theta} \right) + \left(\frac{\cos\theta}{\sin\theta} \right)^2$$

$$= -1$$

$$\equiv + R_{bc}{}^{\phi}{}_{d} V^b N^c V^d$$

$$= + R_{\theta\phi}{}^{\phi}{}_{\theta}$$

So $\quad R_{\theta\phi}{}^{\phi}{}_{\theta} = -1$

Comment : In 2d, Riemann tensor has only 1 independent component (see PS3)