6.1 - Spacetime metric

Spacetime metric is a) Non-degenerate ($\det g \neq 0$)

b) Symmetric (0, 2) tensor

c) Signature $(-,+,+,+)$

Line element:

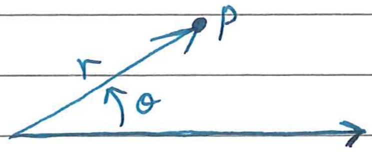
$$ds^2 = g_{ab}(x)\, dx^a dx^b$$

- gives infinitesimal distance between x^a and $x^a + dx^a$.

Example: $\mathbb{R}^2$ + Euclidean metric

$$ds^2 = dx^2 + dy^2$$

$$= dr^2 + r^2 d\theta^2$$



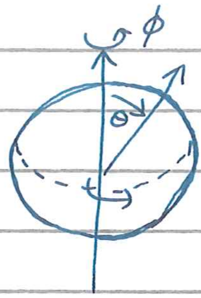
$$\Rightarrow g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

Note: (x, y) are global chart
(r, θ) good for $\mathbb{R}^2 \setminus \{0\}$

Example: S^2 + ~~a~~ round metric

$$0 \leq \phi < 2\pi$$
$$0 < \theta < \pi$$



(Valid away from $\theta = 0, 2\pi$)

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

As for discussion in lectures 2+3

- g_{ab} defines inner product on T_pM

$$g(V, W) = g_{ab} V^a W^b$$

- V^a is TL if $g(V,V) < 0$, etc.

- $g^{ab} g_{bc} = \delta^a{}_c$

- g_{ab} defines $T_pM \cong (T_pM)^*$

$$V_a = g_{ab} V^b \, , \qquad \omega^a = g^{ab} \omega_b$$

6.2 - Levi-Civita covariant derivative

Recall : $D_a \phi = \partial_a \phi$

$$D_a (S^{\cdot\cdot}{}_{\cdot\cdot} + T^{\cdot\cdot}{}_{\cdot\cdot}) = D_a S^{\cdot\cdot}{}_{\cdot\cdot} + D_a T^{\cdot\cdot}{}_{\cdot\cdot}$$

$$D_a (S^{\cdot\cdot}{}_{\cdot\cdot} T^{\cdot\cdot}{}_{\cdot\cdot}) = (D_a S^{\cdot\cdot}{}_{\cdot\cdot}) T^{\cdot\cdot}{}_{\cdot\cdot} + S^{\cdot\cdot}{}_{\cdot\cdot} (D_a T^{\cdot\cdot}{}_{\cdot\cdot})$$

$$D_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c$$

(1,1) tensor — not a tensor

The metric g_{ab} determines a unique connection D_a

* 1) $D_a g_{bc} = 0$ "metric compatible"

2) $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$ "torsion free"

Proof : $D_a g_{bc} = \partial_a g_{bc} - \Gamma^d{}_{ab} g_{dc} - \Gamma^d{}_{ac} g_{bd} = 0 \quad (1)$

$D_b g_{ca} = \partial_b g_{ca} - \Gamma^d{}_{bc} g_{da} - \Gamma^d{}_{ba} g_{cd} = 0 \quad (2)$

$D_c g_{ab} = \ldots \quad (3)$

Consider $(2)+(3)-(1)$ and use $g_{ab} = g_{ba}$, $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$

$$\Rightarrow \partial_b g_{ca} + \partial_c g_{ab} - \partial_a g_{bc} = 2\Gamma^d{}_{bc} g_{da}$$

$$\Rightarrow \Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

These $\Gamma^a{}_{bc}$ are known as "Christoffel symbols"

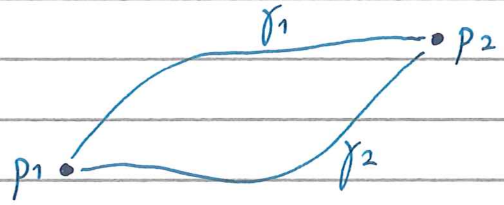
6.3 - Parallel transport

In flat space, $g_{ab} = \eta_{ab}$, can pick $\Gamma^a{}_{bc} = 0$

- Can look at change in a tensor using

$$\partial_a T^b = 0 \Leftrightarrow T^b \text{ constant in } x^a$$

In curved space, $D_a T^b = 0$ is natural generalisation



The way $T^a|_{p_1}$ is transported to p_2 depends on the path taken

- Have to specify the path we are using

For a curve $\gamma = x^a(\lambda)$ with tangent $v^a = \dfrac{dx^a}{d\lambda}$

$$\nabla_v := \frac{D}{D\lambda} = v^a D_a \equiv \nabla_{\dot{\gamma}}$$

<u>Def</u>: A tensor $S^{a_1..a_p}{}_{b_1..b_q}(x)$ is <u>parallel transported</u> along a curve $x^a(\lambda)$ if

$$v^a D_a S^{a_1\ldots}{}_{b_1\ldots} = 0$$

i.e. for v^a tangent to $x^a(\lambda)$

$$D_v S^{a_1\ldots}{}_{b_1\ldots} = 0$$

This specifies a <u>unique</u> way to move a vector along a curve

i.e. given a vector $u^a \in T_p M$ for $p = \gamma(0)$, there is a unique vector field $U^a(\lambda)$ s.t.

1) $D_{\dot\gamma} U^a = 0$

2) $U^a(0) = u^a$

Consider a curve $\gamma(\lambda)$ with $\dot\gamma = T$ that parallel transports its <u>own</u> tangent vector

$$\begin{aligned}
0 &= \cancel{T} D_{\dot\gamma} T^a \\
&= D_T T^a \\
&= T^b D_b T^a \\
&= \frac{dx^b}{d\lambda}\left(\frac{\partial}{\partial x^b}\frac{dx^a}{d\lambda} + \Gamma^a{}_{bc}\frac{dx^c}{d\lambda}\right)
\end{aligned}$$

$$= \frac{d^2 x^a}{d\lambda^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda}$$

$$= \ddot{x}^a + \Gamma^a{}_{bc} \dot{x}^b \dot{x}^c$$

Such a curve $x^a(\lambda)$ is called a <u>geodesic</u>.

<u>Note</u>: we could have taken

$$T^b D_b T^a = \alpha T^a$$

as our defintion.

We can always <u>reparametrise</u> our curve $\lambda \mapsto \lambda'(\lambda)$ such that

$$g_{ab} T^a T^b = T_a T^a = \text{const. as fn. } \lambda$$

Thus

$$\frac{d}{d\lambda}(T^a T_a) = 0$$

$$= \frac{dx^b}{d\lambda} \partial_b (T^a T_a)$$

$\partial \equiv D$ on scalars

$$= T^b D_b (T^a T_a)$$

$$= 2\underbrace{(T^b D_b T^a)}_{\alpha T^a} T_a$$

$$= 2\alpha T^a T_a$$

$$\Rightarrow \quad \alpha = 0$$

A parametrisation in which $T_a T^a = \text{const}$ is called <u>affine</u>.

- The <u>geodesic equation</u> is

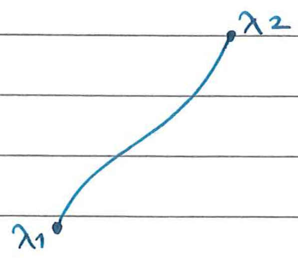
$$T^a D_a T^b = 0$$

- <u>Affine parameters</u> related by

$$\lambda = a\lambda' + b$$

Timelike geodesics minimise <u>proper time</u> τ:

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-g_{ab}(x)\,\dot{x}^a \dot{x}^b}$$



$$\dot{x}^a = \frac{dx^a}{d\lambda}$$