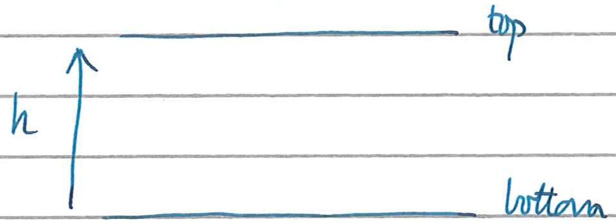
5.1 - Review

In absence of gravity, spacetime is

- $M_4 = \mathbb{R}^4$ as topological space
- Flat metric of signature $(-,+,+,+)$
- There are distinguished inertial frames in which $g = \eta = \mathrm{diag}(-1,1,1,1)$

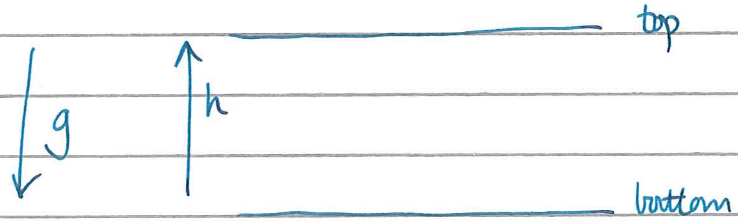
5.2 - Gravity (intuition)

Observers at rest in a global inertial frame agree on time differences



$$\Delta t_t = \Delta t_b$$

Observers at rest in a gravitational field experience time dilation



$$\frac{\Delta t_b}{\Delta t_t} \simeq 1 - gh$$

Can eliminate local effects by moving to a freely falling frame.

5.3 - Gravity (concretely)

There are no global inertial frames in a (non-uniform) gravitational field.

A freely falling observer may set up a local inertial frame in which $g = \eta$

Spacetime is a smooth, 4-dimensional manifold with a metric g_{ab} of signature $(-,+,+,+)$

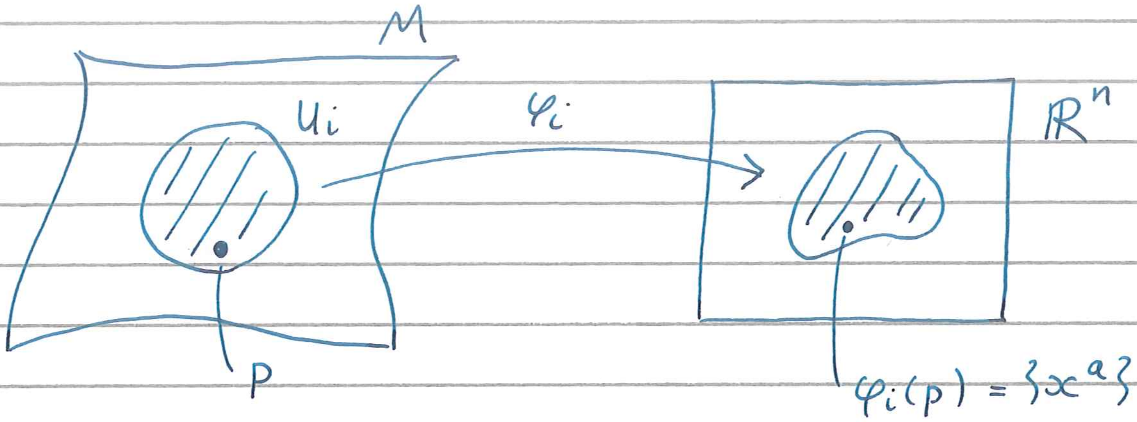
5.4 - Smooth manifolds

Def: 1) M is a topological space (set of points with neighbourhoods satisfying some axioms)

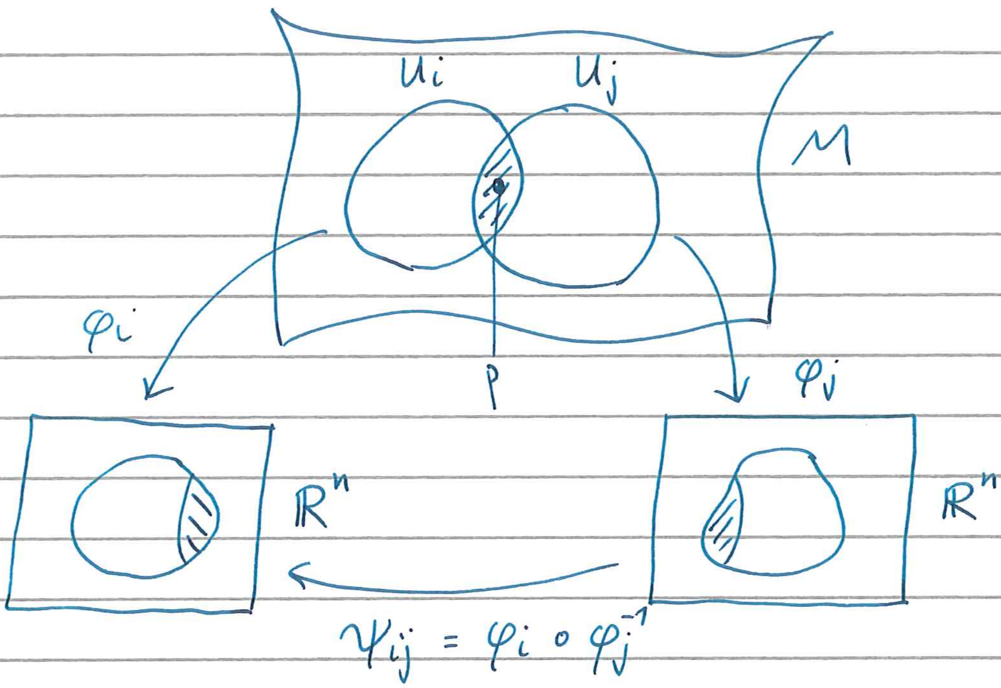
2) M has coordinate charts $\{U_i, \varphi_i\}$ (i labels the chart)

- $U_i \subset M$ are open sets such that $\bigcup_i U_i = M$

- $\varphi_i : U_i \to \mathbb{R}^n$, homeomorphism
  - bijection
  - continuous
  - continuous inverse



3) Compatibility on overlaps



If $\varphi_i(p) = \{x^a\}$

$\varphi_j(p) = \{x'^a\}$ $\quad p \in U_i \cap U_j$

$x'^a = x'^a(x)$ are C^∞

These are <u>general</u> coordinate transformations!

- Might need multiple sets of coordinates / patches to cover manifold.

- If patches fully overlap, just a redefinition of your coordinates

5.5 - Tensor fields

On overlap $U_i \cap U_j$ s.t. $\varphi_i(p) = \{x^a\}$
$\varphi_j(p) = \{x'^a\}$

Example: scalar $f : M \to \mathbb{R}$

$$f(x) \mapsto f(x') = f(x'(x))$$

Example: tangent vector to curve $x^a(\lambda)$

$$T^a(\lambda) := \frac{dx^a}{d\lambda} \mapsto \frac{dx'^a}{d\lambda} = \frac{\partial x'^a}{\partial x^b} \frac{dx^b}{d\lambda}$$

$$= \frac{\partial x'^a}{\partial x^b} T^b(\lambda)$$

Example: gradient of scalar f

$$\partial_a f = \frac{\partial}{\partial x^a} f \mapsto \frac{\partial}{\partial x'^a} f = \frac{\partial x^b}{\partial x'^a} \frac{\partial}{\partial x^b} f$$

$$= \frac{\partial x^b}{\partial x'^a} \partial_b f$$

Alternatively: tensor C^∞ linear in arguments.
$\omega(V) := \omega_a V^a$, $\omega(fV) = f\,\omega(V)$

Important: $\frac{\partial x'^a}{\partial x^b}$ not constant in general!

Example: (p,q) tensor $T^{a_1 \dots a_p}{}_{b_1 \dots b_q}$

$$T^{a_1 \dots a_p}{}_{b_1 \dots b_p} \mapsto T'^{a_1 \dots a_p}{}_{b_1 \dots b_p}(x')$$

$$= \frac{\partial x'^{a_1}}{\partial x^{c_1}} \cdots \frac{\partial x^{d_1}}{\partial x'^{b_1}} \cdots T^{c_1 \dots}{}_{d_1 \dots}(x)$$

As before, we have

- Sum of (p,q) tensors is (p,q)
- Product of (p,q) and (r,s) is $(p+r, q+s)$
- Contraction, $T^{a_1 \dots a_{p-1} c}{}_{b_1 \dots b_{q-1} c}$ is $(p-1, q-1)$

5.6 - Covariant derivatives

Partial derivatives of a tensor do not give another tensor (except for scalars!)

Example: partial derivative of a vector field

$$\frac{\partial V'^a}{\partial x'^b} = \frac{\partial x^c}{\partial x'^b} \frac{\partial}{\partial x^c} \left(\frac{\partial x'^a}{\partial x^d} V^d \right)$$

$$= \underbrace{\frac{\partial x^c}{\partial x'^b}\frac{\partial x'^a}{\partial x^d}\frac{\partial V^d}{\partial x^c}}_{\text{looks right for (1,1) tensor}} + \frac{\partial x^c}{\partial x'^b}\underbrace{\frac{\partial^2 x'^a}{\partial x^c \partial x^d}}_{\text{Non-zero unless } x'^a = L^a{}_b x^b} V^d$$

constant!

Now define <u>covariant derivative</u>

$$D_b V^a = \partial_b V^a + \Gamma^a{}_{bc} V^c$$

- Want $D_b V^a$ to transform as <u>(1,1) tensor</u>.
- $\partial_b V^a$ has extra term so is not a tensor.
- $\Gamma^a{}_{bc}$ defined so that it <u>cancels</u> this extra piece!

$$\Gamma'^a{}_{bc} = \frac{\partial x^p}{\partial x'^b}\frac{\partial x^q}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^r}\Gamma^r{}_{pq} - \frac{\partial^2 x'^a}{\partial x^p \partial x^q}\right)$$

<u>Exercise</u> on PS2 - check this!

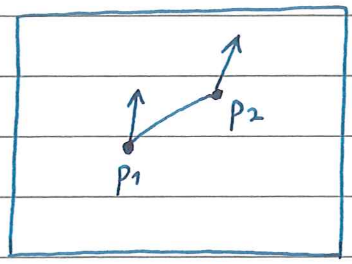
- Check $D_b V^a$ then transforms as (1,1) tensor

$$D_b V^a \mapsto D'_b V'^a = \frac{\partial x^c}{\partial x'^b}\frac{\partial x'^a}{\partial x^d} D_c V^d$$

What is <u>really</u> going on?

Derivative lets you compare a tensor at <u>different</u> points on a manifold

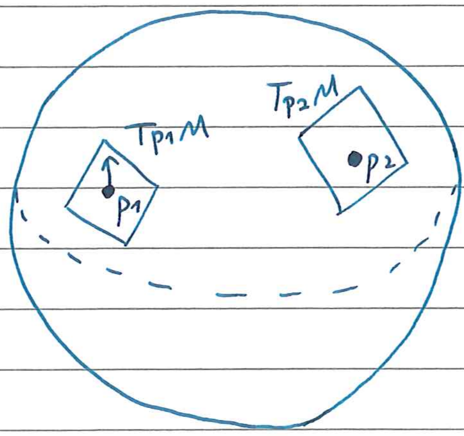
e.g. For $\mathbb{R}^2$



<u>Canonical</u> way to transport vector at p_1 to p_2 to compare (just use ∂_a). $T_{p_1}\mathbb{R}^2 = T_{p_2}\mathbb{R}^2$

Not so simple for curved manifolds.

e.g. for S^2



Prescription for transporting vector from p_1 to p_2 needs <u>extra</u> information.

$T_{p_1}S^2 \cong T_{p_2}S^2$ but no <u>canonical</u> identification

Extra input is a connection ∇_a - allows us to compare vectors in different <u>tangent</u> <u>spaces</u>.

Covariant derivative ∇_a extends to (p,q) tensors as

- Linearity $\nabla_a (S^{b_1 \dots}{}_{c_1 \dots} + T^{b_1 \dots}{}_{c_1 \dots})$
$$= \nabla_a S^{b_1 \dots}{}_{c_1 \dots} + \nabla_a T^{b_1 \dots}{}_{c_1 \dots}$$

- Products $\nabla_a (S^{b_1 \dots}{}_{c_1 \dots} T^{d_1 \dots}{}_{e_1 \dots})$
$$= (\nabla_a S^{b_1 \dots}{}_{c_1 \dots}) T^{d_1 \dots}{}_{e_1 \dots} + S^{b_1 \dots}{}_{c_1 \dots} (\nabla_a T^{d_1 \dots}{}_{e_1 \dots})$$

- Action on scalars $\nabla_a f = \partial_a f$

Example : $\phi = v^a \omega_a$

$$\nabla_a \phi := \partial_a \phi = (\partial_a v^b) \omega_b + v^b (\partial_a \omega_b)$$

but
$$= \nabla_a (v^b \omega_b)$$
$$= (\nabla_a v^b) \omega_b + v^b (\nabla_a \omega_b)$$
$$= (\partial_a v^b + \Gamma^b{}_{ac} v^c) \omega_b + v^b (\nabla_a \omega_b)$$

Holds for any v^b, so

$$\nabla_a \omega_b = \partial_a \omega_b - \Gamma^c{}_{ab} \omega_c$$

Can find action on (p,q) tensor by induction

$$D_a T^{b_1 .. b_p}{}_{c_1 .. c_p} = \partial_a T^{b_1 ..}{}_{c_1 ..}$$
$$+ \Gamma^{b_1}{}_{ab} T^{b b_2 .. b_p}{}_{c_1 ..} + \Gamma^{b_2}{}_{ab} T^{b_1 b b_3 ..}{}_{c_1 ..}$$
$$- \Gamma^{c}{}_{ac_1} T^{b_1 ..}{}_{c c_2 ..} - \Gamma^{c}{}_{ac_2} T^{b_1 ..}{}_{c_1 c c_3 ..}$$

Can think of $\Gamma^c{}_{ab}$ as $(\Gamma_a)^c{}_b$

- For each vector index, $(\Gamma_a \cdot v)^c = (\Gamma_a)^c{}_b v^b$
$$= \Gamma^c{}_{ab} v^b$$
- For each 1-form index, $(\Gamma_a \cdot \omega)_b = -(\Gamma_a)^c{}_b \omega_c$
$$= -\Gamma^c{}_{ab} \omega_c$$

$\Gamma^a{}_{bc}$ (and D_a) are <u>not</u> unique for a given manifold

$$\Gamma^a{}_{bc} \mapsto \hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + Q^a{}_{bc}$$

is equally fine if $Q^a{}_{bc}$ is an honest $(1,2)$ tensor

- Have connections that differ by $Q^a{}_{bc}$.
- How to fix this ambiguity?

Physics: Spacetime is a manifold + a metric

- Want to measure distances
- Want ∇_a and metric to be compatible

Peak ahead: **Unique** ∇_a such that

1) $\nabla_a g_{bc} = 0$

2) $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$

This is the connection that appears in GR.