Note that $\frac{d}{d\tau}(u^\mu u^\nu \eta_{\mu\nu}) = 0$

$$= 2 a^\mu u^\nu \eta_{\mu\nu}$$

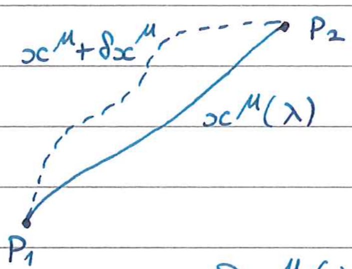
$$\Rightarrow \quad \eta(a, u) = 0$$

If $a^\mu \neq 0$, a^μ orthogonal to u^μ

$$\Rightarrow \quad a_\mu a^\mu > 0 \qquad \text{"spacelike"}$$

Action and proper time

Physical trajectories extremise proper time



$$\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$$

Action for free particle

$$S[x] = -m \int d\tau$$

$$= -m \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

With a parametrisation $x^\mu(\lambda)$

$$S[x] = \int d\lambda \, \mathcal{L}_\lambda$$

$$\mathcal{L}_\lambda = -m\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

e.g. $\lambda = x^0 = t$

$$\mathcal{L}_t = -m\sqrt{1-|\vec{v}|^2}$$

$$\vec{v} = \frac{d\vec{x}}{dt}$$

Can always choose λ s.t.

$$\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} = \text{constant}$$

λ is then an "affine parameter"
$\lambda = a\tau + b$ for τ proper time.

Important: for a timelike curve $x^\mu(\lambda)$, extremising $S[x]$ is equivalent to extremising

$$S[x] = \int d\lambda \, \frac{1}{2}\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

This simpler functional can also be used for massless particles!

Curves / trajectories / worldlines that extremise the action are known as "geodesics".

- Analogue of straight lines in M_4.

4 - momentum

Defined as $p^\mu = m u^\mu = (E, \vec{p})$

$$p_\mu p^\mu = -m^2 \qquad \text{"mass shell relation"}$$

rest mass

Given a reference frame / coordinates in which

- particle has momentum p^μ
- observer $\mathcal{O}$ has velocity v^μ

Observer will measure energy of particle to be

$$E_{\mathcal{O}} = -\eta_{\mu\nu} p^\mu v^\nu$$

e.g. Particle at rest w.r.t. $\mathcal{O}$

- $p^\mu = m v^\mu$

$E_{\mathcal{O}} = m$ "rest mass in $\mathcal{O}$ frame"

Can work in frame of $\mathcal{O}$ $\left(p^\mu = (m, \vec{0}),\ v^\mu = (1, 0) \right)$
or in a different frame $\left(v^\mu = \gamma(1, \vec{v}),\ p^\mu = m v^\mu \right)$.

e.g. Particle moving at v in $\hat{x}$ direction in $\mathcal{O}$ frame

$$v^\mu = (1, \vec{0}) \qquad \text{"}\mathcal{O} \text{ at rest in own frame"}$$

$$p^\mu = \gamma m (1, v, 0, 0)$$

$$\Rightarrow E_{\mathcal{O}} = m\gamma \cong \underbrace{m}_{\text{rest mass}} + \underbrace{\tfrac{1}{2} v^2}_{\text{kinetic energy}} + \ldots$$

Stress - energy tensor

Energy density, energy flux, momentum density and pressure encoded by a symmetric $(2,0)$ tensor $T^{\mu\nu}$

Let $T^{\mu\nu}$ be stress - tensor and u^μ the 4 - velocity of a timelike observer $\mathcal{O}$ in an inertial frame

- $\mathcal{O}$ measures a 4 - momentum density

$$j^\mu = - T^{\mu\nu} u_\nu$$

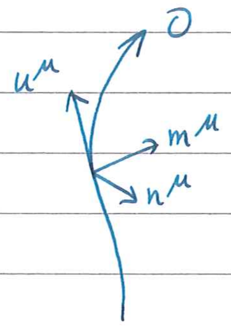
- Energy is $- u^\mu p_\mu$, so energy density is

$$\rho = - u^\mu j_\mu = T_{\mu\nu} u^\mu u^\nu$$

- for normal matter, $T_{\mu\nu} u^\mu u^\nu \geq 0$, "<u>weak energy condition</u>".

Let m^μ and n^μ be spacelike vectors s.t.

- $m_\mu m^\mu = n_\mu n^\mu = +1$
- $u_\mu m^\mu = u_\mu n^\mu = 0$

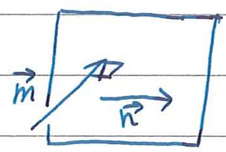


<u>Pressure</u> in m^μ direction measured by $\mathcal{O}$ is

$$p = T_{\mu\nu} m^\mu m^\nu$$

<u>Stress</u> in n^μ direction across surface $\perp$ to m^μ is

$$S = T_{\mu\nu} m^\mu n^\nu$$



<u>Conservation</u> of energy and momentum is

$$\partial_\mu T^{\mu\nu} = 0$$

- Can derive this from Noether's theorem for translations
- For a field theory coupled to gravity

with action $S[\phi, g_{ab}]$

$$T_{ab} = \frac{\delta S}{\delta g^{ab}}$$

Perfect fluid

Macroscopic description of matter in terms of pressure and density.

e.g. gases or fluids.

A "perfect fluid" is one such that an observer in the rest frame of the fluid sees it as isotropic

- "comoving" observer sees it as rotation invariant.

In reference frame of fluid

$$T_{00} = \rho \,, \quad T_{ij} = p\,\delta_{ij} \,, \quad T_{0i} = 0$$

- ρ : "energy density".
- p : "pressure".

To specify the kind of fluid, give a relation between ρ and p

"<u>Equation of state</u>": $p = p(\rho)$

In an inertial frame where fluid moves with 4-velocity u^μ

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\,\eta_{\mu\nu}$$

e.g. for $u^\mu = (1, \vec{0})$, $\quad T_{\mu\nu} u^\mu u^\nu = \rho$

"<u>Dust</u>" is pressureless ($p = 0$, particles do not bump into each other much) - project along u^μ for energy conservation.

$$u_\mu \partial_\nu T^{\nu\mu} = u_\mu \partial_\nu (\rho u^\nu u^\mu)$$

$$= u_\mu u^\mu \underbrace{\partial_\nu (\rho u^\nu)}_{j^\nu} + \rho u^\nu \underbrace{u_\mu \partial_\nu u^\mu}_{0 \text{ as } \partial_\nu (u_\mu u^\mu) = 0}$$

$$= - \partial_\nu j^\nu$$

"Fluid density conserved".

In $|v| \ll 1$ limit, $u^\mu = \gamma(1, \vec{v}) \simeq (1, \vec{v})$

$$j^\mu = \rho u^\mu \cong (\rho, \rho\vec{v}),$$

$$\partial_\mu j^\mu = \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$ "conservation of mass" "continuity equation"

4.4 - Electromagnetic flux

Electromagnetic field encoded in a $(0,2)$ anti-symmetric tensor F_{ab}

- $F_{\mu\nu}$ has 6 components (3 electric, 3 magnetic)

An observer with 4-velocity v^μ measures

- Electric field : $E_\mu = F_{\mu\nu} v^\nu$

  E^μ is spacelike with 3 independent components (see PS1)

- Magnetic field : $B_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\nu\rho} v^\sigma$

  where $\epsilon_{0123} = +1$ and $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{[\mu\nu\rho\sigma]}$

  Again, B^μ spacelike with 3 components.

Example : O at rest, $v^\mu = (1, \underline{0})$

$$E_\mu = F_{\mu 0} \quad \Rightarrow \quad E_i = F_{i0}$$

$$B_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho 0} F^{\nu\rho}$$

$$= +\frac{1}{2} \epsilon_{0\mu\nu\rho} F^{\nu\rho}$$

$$\Rightarrow \quad B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$$

$$\Rightarrow \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

Example : $\mathcal{O}$ with $v^\mu = \gamma(v)(1, \underline{v})$

$$E'_i = F_{i\mu} v^\mu$$

$$= \gamma(v)\left(F_{i0} + F_{ij} v^j \right)$$

$$= \gamma(v)\left(E_i + \tfrac{1}{2} \epsilon_{ijk} v^j B^k \right)$$

$$\Rightarrow \quad \vec{E}' = \gamma(v)\left(\vec{E} + \vec{v} \times \vec{B} \right)$$

Electric vs. magnetic is dependent on the inertial frame.