Mathematical Physiology

Problem sheet 4

Answers

Warning: the large α part of this is slightly inaccurate

1. $\dot{x} = \alpha x(1-x_1)$

Linearise about $x=1$, $x = 1+X$,

$$\dot{X} = \alpha(1+X).-X_1 \simeq -\alpha X_1$$

Solutions $X = e^{\sigma t}$

$$\sigma = -\alpha e^{-\sigma}$$

i if σ is real, then $\sigma < 0$ (as $\alpha > 0$)

ii Consideration of the essential singularity of σe^{σ} at ∞

$\Rightarrow$ (Picard's theorem) $\exists\ \infty$ # roots (since the excluded value near ∞ is 0, thus $\sigma e^{\sigma} = -\alpha$ infinitely often)

iii Suppose $\operatorname{Re}\sigma > 0$ and α is small:

then $|\sigma| = \alpha|e^{-\sigma}| < \alpha$ is small

therefore $\sigma = -\alpha e^{\sigma}$

$= -\alpha[1+O(\alpha)]$

So $\sigma = -\alpha + O(\alpha^2)$ & $\operatorname{Re}\sigma < 0$ ✗

therefore $\operatorname{Re}\sigma < 0$ if α is small enough.

iv σ varies smoothly (actually analytically) with α

Since $\sigma e^{\sigma} = -\alpha \Rightarrow e^{\sigma}(1+\sigma)\sigma' = -1$ $\quad (\sigma' = \frac{d\sigma}{d\alpha})$

thus $\sigma' = \dfrac{-e^{-\sigma}}{1+\sigma} = \dfrac{\sigma}{\alpha(1+\sigma)}$ with poles at $\sigma = -1$ (in the complex α plane)

[redacted]

v Transversality: if $\sigma = i\omega$ for some α, then

$$\sigma' = \frac{i\omega}{\alpha(1+i\omega)} = \frac{i\omega(1-i\omega)}{\alpha(1+\omega^2)}$$

So $\operatorname{Re}\sigma' = \dfrac{\omega^2}{\alpha(1+\omega^2)} > 0$

Hence $\exists \operatorname{Re}\sigma > 0$ for $\alpha > \alpha_c$ if at $\alpha = \alpha_c$ $\sigma = \pm i\omega$

wz (eg $\sigma = i\omega$) $\quad i\omega = -\alpha e^{-i\omega} = -\alpha\left[\cos\omega - i\sin\omega\right]$

$$\Rightarrow \quad 0 = -\alpha\cos\omega$$
$$\omega = \alpha\sin\omega$$

The first of these $\Rightarrow \omega = \pi/2,\ 3\pi/2$ etc (wlog $\omega > 0$)

& the corresponding values of $\alpha = \pi/2,\ -3\pi/2,\ +5\pi/2,\ -7\pi/2$

& for $\alpha > 0$ we have $\alpha_c = \pi/2$

Suppose $x = e^{\alpha t}$ for $t < 0$.

Let $\phi = \ln x$, $\phi = \alpha t$, $t < 0$

& $\dot{\phi} = \alpha[1 - \phi_1]$

0<t<1

$$\phi_1 = \alpha(t-1)$$

$$\dot{\phi} = \alpha - \alpha e^{\alpha(t-1)} \quad , \phi = 0, \ t = 0$$

$$\Rightarrow \phi = \alpha t - e^{\alpha(t-1)} - e^{-\alpha}$$

i.e. $\phi \approx \alpha t - e^{\alpha(t-1)}$, $0 < t < 1$ (as $e^{-\alpha} \ll 1$)

1<t<2

$$\phi_1 = \alpha(t-1) - e^{\alpha(t-2)}$$

$$\Rightarrow \dot{\phi} = \alpha - \alpha e^{\alpha(t-1)} e^{-e^{\alpha(t-2)}}$$

& $\phi = \alpha - 1$ at $t = 1$

Note $\dot{\left(e^{-e^{\alpha(t-2)}}\right)} = -\alpha e^{\alpha(t-2)} e^{-e^{\alpha(t-2)}}$

So $\dot{\phi} = \alpha - \alpha e^{\alpha} e^{\alpha(t-2)} e^{-e^{\alpha(t-2)}}$

& thus $\phi = \alpha(t-1) + e^{\alpha} e^{-e^{\alpha(t-2)}} - e^{\alpha} e^{-e^{-\alpha}} + \alpha - 1$

$$= \alpha(t-1) + e^{\alpha} e^{-e^{\alpha(t-2)}} - e^{\alpha}[1 - e^{-\alpha} \ldots] + \alpha - 1$$

$$= \alpha t + e^{\alpha} e^{-e^{\alpha(t-2)}} - e^{\alpha} + 1 \ldots - 1$$

i.e. $\phi \simeq \alpha t + e^{\alpha}\left[e^{-e^{\alpha(t-2)}} - 1\right]$

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~~and~~

<u>$2 < t < 3$</u>

$$\phi_1 \simeq \alpha(t-1) + e^{\alpha}\left[e^{-e^{\alpha(t-3)}} - 1\right]$$

Now so long as $t < 3$ & $e^{\alpha(t-3)} \ll 1$ (& t not close to 3)

& $\phi_1 \simeq \alpha(t-1) + e^{\alpha}\left[1 - e^{\alpha(t-3)} \cdots - 1\right]$

[note: to neglect the next (+ higher terms) we need $t < 2.5$ which is still ok]

$$\Rightarrow \phi_1 \simeq \alpha(t-1) - e^{\alpha(t-2)}$$

This is the same as for $1 < t < 2$! Thus same solution, except

$$\phi \simeq 2\alpha - (1 - e^{-1})e^{\alpha} \quad \text{at } t = 2$$

$$\Rightarrow \phi = 2\alpha - (1-e^{-1})e^{\alpha} + \alpha(t-2) + e^{\alpha - e^{\alpha(t-2)}} - e^{\alpha - 1}$$

$$= \alpha t - e^{\alpha}\left[1 - e^{-e^{\alpha(t-2)}}\right]$$

same as $t < 2$!

In summary,

$$\phi \simeq \alpha t, \quad t < 0$$

$$\phi \simeq \alpha t - e^{\alpha(t-1)}, \quad 0 < t < 1$$

$$\phi \simeq \alpha t + e^{\alpha}\left[e^{-e^{\alpha(t-2)}} - 1\right] \quad 1 < t < 2$$

$$\phi \simeq \alpha t + e^{\alpha}\left[e^{-e^{\alpha(t-2)}} - 1\right] \quad 2 < t \lesssim 2.5$$

Clearly ϕ is increasing in $t<0$ & also in $0<t<1$

But for $1<t<2$ (& away from $t=2$)

$$\phi \simeq \alpha t + e^{\alpha}\left[1 - e^{\alpha(t-2)} \ldots - 1\right]$$

$$\simeq \alpha t - e^{\alpha(t-1)}$$

so reaches a maximum and then rapidly decreases

At $t=2$, ϕ is large and negative, but for $t>2$

~~$\phi \simeq \alpha t + e^{\alpha}[$~~ $e^{\alpha(t-2)}$ becomes large

& so $\phi \simeq \alpha t - e^{\alpha}$ becomes an increasing function again

and the cycle repeats. If the period is P, then

$$\phi \simeq \alpha(t-P) = \alpha t - e^{\alpha}$$

$$\Rightarrow P \simeq \frac{e^{\alpha}}{\alpha}$$

To find the maximum and minimum,

for $t > 1$ $\quad \phi \simeq \alpha t - e^{\alpha(t-1)}$

max at $\quad \alpha = \alpha e^{\alpha(t-1)} \quad \Rightarrow t = 1$: not good enough, so

revert to

$$\phi \simeq \alpha t + e^{\alpha}\left[e^{-e^{\alpha(t-2)}} - 1\right]$$

$$\Rightarrow \dot{\phi} = \alpha - \alpha e^{\alpha} e^{\alpha(t-2)} e^{-e^{\alpha(t-2)}}$$

Put $\alpha(t-1) = \xi$ $\qquad$ [so $\alpha(t-2) = \alpha(t-1) - \alpha$ $= \xi - \alpha$]

$\dot{\phi} = 0$ if $\quad 1 = e^{\xi} e^{-e^{\xi - \alpha}}$

viz $\quad e^{\xi - \alpha} = \xi \quad \Rightarrow \xi e^{-\xi} = e^{-\alpha}$

There are two roots, $\xi \sim \alpha \quad \Rightarrow t \simeq 2$ ✗.

and $\xi \sim e^{-\alpha} \ll 1$

giving $t_{max} \simeq 1 + \dfrac{e^{-\alpha}}{\alpha}$

and $\phi_{max} \simeq \alpha + \xi - e^{\xi} \simeq \alpha - 1$

& $\eta_{max} \simeq e^{\alpha - 1}$

To find $t_{min} > 2$

use $\phi \simeq \alpha t - e^{\alpha}\left[1 - e^{-e^{\alpha(t-2)}}\right]$

$\Rightarrow \dot{\phi} = \alpha - e^{\alpha} \alpha e^{\alpha(t-2)} e^{-e^{\alpha(t-2)}}$

$= 0$ if $\quad \alpha + \alpha(t-2) - e^{\alpha(t-2)} = 0$

~~case~~ Define $\alpha(t-2) = \eta$

$\Rightarrow \quad \alpha + \eta = e^{\eta}$

$\Rightarrow \quad \eta = \ln\alpha + \ln\left(1 + \frac{\eta}{\alpha}\right)$

$\eta \simeq \ln\alpha + \frac{\ln\alpha}{\alpha} \ldots$

& $t_{min} \simeq 2 + \frac{\eta}{\alpha} \simeq 2 + \frac{\ln\alpha}{\alpha} \ldots$

and $\phi_{min} \simeq 2\alpha + \eta - e^{\alpha}\left[1 - e^{-e^{\eta}}\right]$

$= 2\alpha + \eta - e^{\alpha} + e^{\alpha - (\alpha+\eta)}$

$= 2\alpha + \eta - e^{\alpha} + \frac{1}{\alpha+\eta}$

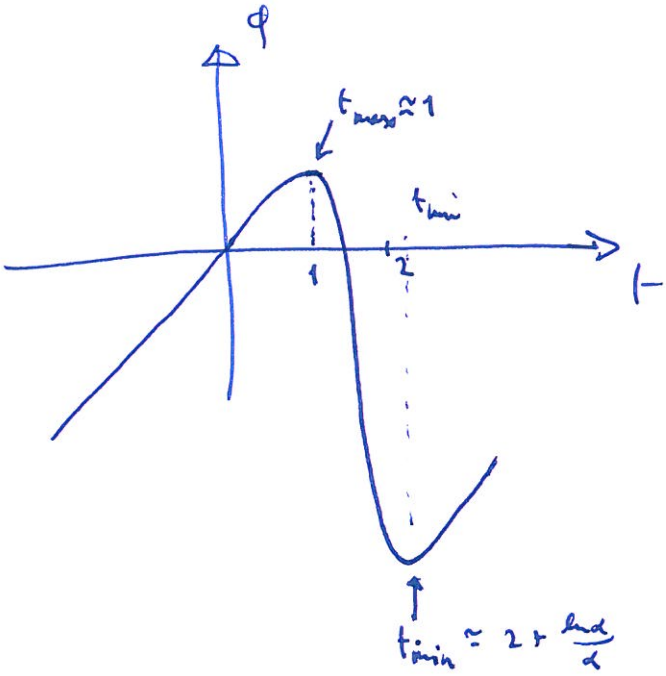
$\simeq -e^{\alpha} + 2\alpha + \eta + o(1)$

& $x_{min} \simeq e^{\eta} e^{2\alpha - e^{\alpha}} \ldots$

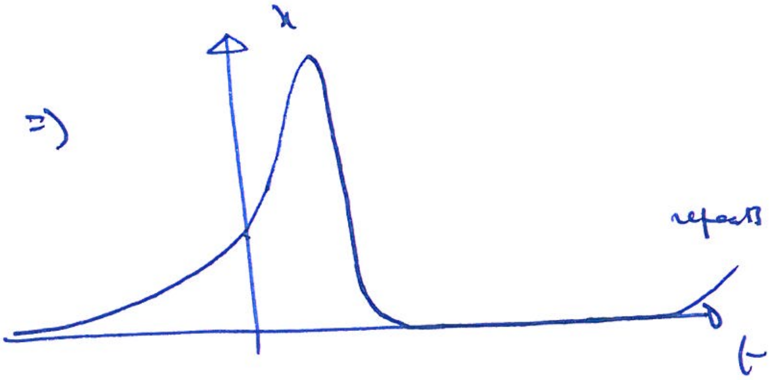
$\simeq (\alpha + \eta) e^{2\alpha - e^{\alpha}}$

$\simeq \alpha e^{2\alpha - e^{\alpha}}$

We have



$\Rightarrow$



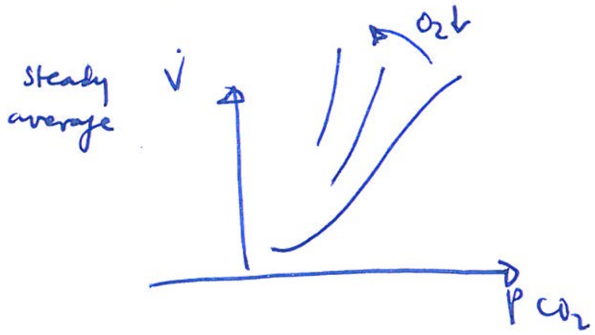
Solution is good for $\alpha = 3$ as $\min x \sim e^{-e^{\alpha}}$ is small for $\alpha > 2$!!

2. Minute ventilation:

volume of air breathed in a minute

Central chemoreceptor responds to H^+ & thus effectively to CO_2 via the bicarbonate buffering system.

Peripheral chemoreceptors also respond to CO_2 but the response is amplified if O_2 is lowered



Mackey–Glass

$$K\dot{p} = M - p\dot{V}$$
$$\dot{V} = \dot{V}(p_\tau)$$

K tissue volume

M metabolic production rate

$\dot{V}$ ventilation

1st eq conservation of CO_2

2nd controller

$$\dot V = G[p - p_0]_+$$

$M = 200$ mm Hg l (BTPS) min^{-1}

$p_0 = 35$ mm Hg

$K = 40$ l (BTPS)

$G = 2$ l (BTPS) min^{-1} mm Hg^{-1}

$\tau = 0.2$ min

$$K\dot p = M - p\,\overbrace{G(p_\tau - p_0)_+}^{\dot V}$$

Define
$$p = p_0 + (\Delta p)p^*$$
$$\dot V = G\Delta p\, v \qquad \Rightarrow v = [p_1^*]_+$$
$$t \sim \tau$$

and (drop $*$)
$$\frac{K\Delta p}{\tau}\dot p = M - p_0\left(1 + \frac{\Delta p}{p_0}p\right)G\Delta p\, v$$

Choose $M = p_0 G\Delta p$ i.e. $\Delta p = \dfrac{M}{p_0 G} \sim \dfrac{200 \text{ mm Hg l min}^{-1}}{35 \text{ mm Hg } 2 \text{ l min}^{-1}\text{ mm Hg}^{-1}} \simeq 3$ mm Hg

$$\mu = \frac{\Delta p}{p_0} = \frac{M}{p_0^2 G}$$

then
$$\dot p = \frac{M\tau}{K\Delta p}\left[1 - (1+\mu p)v\right]$$
$$= \alpha\left[1 - (1+\mu p)v\right]$$

$$\alpha = \frac{M\tau}{K\Delta p} = \frac{G p_0 \tau}{K}$$

So $\alpha \sim \dfrac{2 \text{ l min}^{-1}\text{ mm Hg}^{-1}\; 35 \text{ mm Hg}\; 0.2 \text{ min}}{40 \text{ l}}$

$\simeq \dfrac{1}{3}$ & $\mu \simeq \dfrac{3}{35} \simeq 0.08$

Conversion factor from wet saturated body temperature to dry NTP : $863 = \frac{760 \times 310}{273}$

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venous arterial

$$K_L \dot{P}_{aCO_2} = -\dot{V} P_{aCO_2} + 863\, K_{CO_2} Q \left[P_{vCO_2} - P_{aCO_2} \right] \quad (1)$$

lung volume — ventilatory loss — solubility coefficient — blood flow

$$K_{CO_2} K_B \dot{P}_{BCO_2} = MR_{TBCO_2} + K_{CO_2} Q_B \left[P_{aCO_2}(t-\tau_{aB}) - P_{BCO_2} \right] \quad (2)$$

brain volume — production rate in [redacted] brain — brain blood flow — arterial delay to brain

$$K_{CO_2} K_T \dot{P}_{TCO_2} = MR_{TCO_2} + (Q-Q_B) K_{CO_2} \left[P_{aCO_2}(t-\tau_{aT}) - P_{TCO_2} \right] \quad (3)$$

tissue volume — tissue production rate — delay to tissues

$$Q P_{vCO_2} = Q_B P_{BCO_2}(t-\tau_{vB}) + (Q-Q_B)\, P_{aCO_2}(t-\tau_{vT}) \quad (4)$$

delay from tissues

(1) : Conservation of arterial/lung CO_2

(2) : " brain CO_2

(3) : " tissue CO_2

(4) : venous blood flow CO_2 is the sum of the flow from the brain & that from other tissues

Response time scales (green terms)

(1) $t_a = \dfrac{K_L}{863 K_{CO_2} Q}$ (since $863 K_{CO_2} Q \gg \dot{V}$) $\sim \frac{3}{26} \simeq 0.12$ min

(2) $t_B = \dfrac{K_B}{Q_B} \sim \dfrac{1}{0.75} = \frac{4}{3}$ min $= 80$ s

(3) $t_T = \dfrac{K_T}{Q} \sim \dfrac{39}{6} \sim 6.5$ min

Therefore on a time scale of a minute P_{aCO_2} is rapid →

P_{aCO_2} → quasi-equilibrium.

t_T is slow so P_{TCO_2} changes slowly

$$(*) \Rightarrow \quad P_{vCO_2} = P_{TCO_2} + \frac{Q_B}{Q}\left[P_{BCO_2} - P_{TCO_2}\right]$$

and since $\frac{Q_B}{Q} \simeq 0.15$,

$$P_{vCO_2} \simeq P_{TCO_2} \text{ is slowly varying}$$

but also
$$-\frac{\dot{V}}{863 K_{CO_2} Q} P_{aCO_2} + P_{vCO_2} - P_{aCO_2} \simeq 0$$

$$\Rightarrow \quad P_{aCO_2} \simeq \frac{P_{vCO_2}}{1 + \frac{\dot{V}}{863 K_{CO_2} Q}} \simeq \frac{P_{TCO_2}}{1 + \frac{\dot{V}}{863 K_{CO_2} Q}}$$ gives P_{aCO_2}

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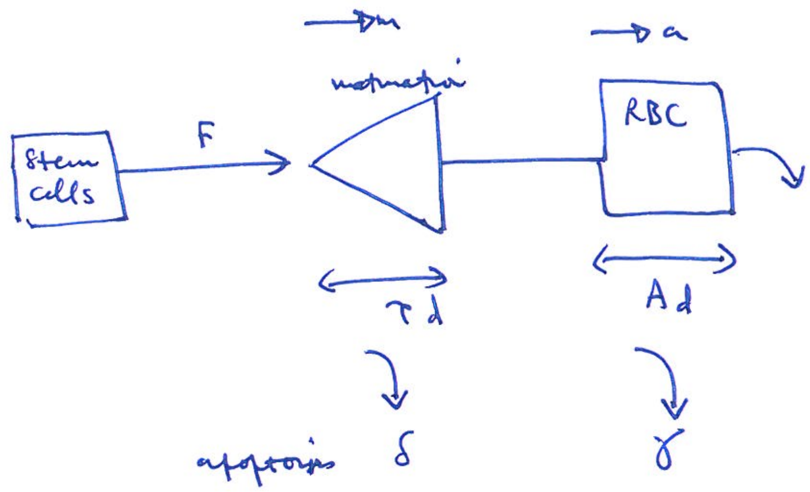


Let $p(t, m)$ be the proliferative cell density (m is maturation time)

$$\Rightarrow \frac{\partial p}{\partial t} + \frac{\partial p}{\partial m} = -\delta p, \quad 0 < m < \tau$$

Let $e(t, a)$ be circulating RBC density as function of age a

$$\Rightarrow \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} = -\gamma e, \quad 0 < a < A$$

Now with $P = \int_0^\tau p \, dm$

we have $\frac{dP}{dt} + p|_{m=\tau} - p|_{m=0} = -\delta P$

ie $\frac{dP}{dt} = p|_{m=0} - \delta P - p|_{m=\tau}$

clearly $p|_{m=0}$ = supply rate $= F$

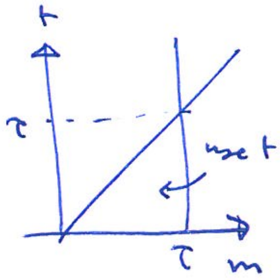
& $p|_{m=\tau}$ = down rate to circulation

(thus $e|_{a=0} = p|_{m=\tau}$)

Characteristics :

$\dot{p} = -\delta p$

$\dot{m} = 1$



use $t=0$, these wash out of system

Use boundary conditions on $m=0$

$m=0$

$t=s$

$p=F(s)$ (specifically $F[E(s)]$, $E(t)=\int_0^A e(t,a)\,da$)

$\Rightarrow \; p = F(s)e^{-\delta(t-s)}$

$m = t-s$

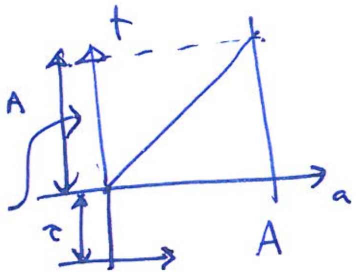
So $p = F(t-m)e^{-\delta m} \quad \forall m$ if $t>\tau$

$\Rightarrow \; p|_{m=\tau} = e|_{a=0} = F(t-\tau)e^{-\delta\tau}$

Next

$\dot{e} = -\gamma e$

$\dot{a} = 1$



$\Leftarrow \left\{ \begin{array}{l} a=0 \\ t=\xi \\ e=F(\xi-\tau)e^{-\delta\tau} \end{array} \right.$

$e = F(\xi-\tau)e^{-\delta\tau}e^{-\gamma(t-\xi)}$

$a = t-\xi$

can only use this for $\xi>\tau$

ie $t>a+\tau$

$\Rightarrow \; e = F(t-a-\tau)e^{-\delta\tau-\gamma a}$

and works for all a if $t > A+\tau$

Thus with $E = \int_0^A e\,da$

$\dot{E} = -\gamma E + e|_{a=0} - e|_{a=A}$

$= -\gamma E + F[E(t-\tau)]e^{-\delta\tau} - F[E(t-A-\tau)]e^{-\delta\tau-\gamma A}$

If there is no age limit, then $A \to \infty$

& just $\dot{E} = -\gamma E + f(E_\tau) e^{-\delta\tau}$

Such reduction occurs if $\gamma A \gg 1$ for example

Non-d $\quad f = F_0 g$

& scale $E \sim E_0 = \dfrac{F_0 e^{-\delta\tau}}{\gamma}$, $t \sim \tau$

then non-d $\quad \dfrac{E_0}{\tau}\dot{E} = -\gamma E_0 E + F_0 g(E_1) e^{-\delta\tau} - F_0 g\left[E\left(t-1-\frac{A}{\tau}\right)\right] e^{-\delta\tau-\gamma A}$

thus $\quad \dfrac{1}{\gamma\tau}\dot{E} = -E + g(E_1) - e^{-\gamma A} g\left[E\left(t-1-\frac{A}{\tau}\right)\right]$

define $\mu = \gamma\tau$, $\Lambda = \dfrac{A}{\tau}$ $\quad \Rightarrow \gamma A = \mu\Lambda$

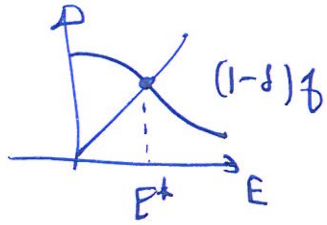
& $\quad \dot{E} = \mu\left[-E + g(E_1) - e^{-\mu\Lambda} g(E_{1+\Lambda})\right]$

$A = 120\,d$ $\quad \tau = 6\,d$, $\Rightarrow \Lambda = 20$

If μ is not small then the finite lifetime is irrelevant – so why would A be so big. Also we would hope E would be relatively constant & this is facilitated by $\mu \ll 1$.

Note $\delta = e^{-\mu\Lambda} < 1$

Steady state $E = (1-\delta)\, f(E)$



f is decreasing $\Rightarrow$ unique steady state

Linearise $E = E^* + \eta$

$$\Rightarrow \dot{\eta} \simeq \mu\left[-\eta + f'\eta_1 - \delta f'\eta_{1+\Lambda}\right]$$

$\eta = e^{\sigma t}$

$$\Rightarrow \sigma = \mu\left[-1 - |f'|\left\{e^{-\sigma} - \delta e^{-\sigma(1+\Lambda)}\right\}\right]$$

$$\Rightarrow \sigma = -\mu\left[1 + |f'|e^{-\sigma}\left\{1 - \delta e^{-\sigma\Lambda}\right\}\right]$$

Suppose that $\mathrm{Re}\,\sigma > 0$:

this requires $|f'|\left|e^{-\sigma}\left\{1-\delta e^{-\sigma\Lambda}\right\}\right| > 1$

but LHS $< |f'|\,|1-\delta e^{-\sigma\Lambda}|$

& since $\delta < 1$, $|e^{-\sigma\Lambda}| < 1$, certainly $|1-\delta e^{-\sigma\Lambda}| < 2$

So LHS $< 2|f'|$

So E^* is stable if $|f'| < \frac{1}{2}$

Since $\Lambda \gg 1$, $|1-\delta e^{-\sigma\Lambda}| \simeq 1$ & this becomes $|f'| < 1$

[unless $\mathrm{Re}\,\sigma \simeq 0$ ....]

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[See also the file question4_5.pdf for graphical illustrations.]

$$\sigma = -\alpha - \Gamma e^{-\sigma}$$

$$\Gamma_0(\alpha) = \frac{\Omega}{\sin\Omega}, \quad \alpha = -\frac{\Omega}{\tan\Omega}$$

$$\text{So } \Gamma_0(\alpha) = \left[1 - \frac{\Omega^2}{6} + \frac{\Omega^4}{120} - \frac{\Omega^6}{120\times 42} + O(\Omega^8)\right]^{-1}$$

$$= \left[1 - \left\{\tfrac{1}{6}\Omega^2 - \tfrac{1}{120}\Omega^4 + \tfrac{1}{120\times 42}\Omega^6\right\} \ldots\right]^{-1}$$

$$\begin{aligned}
&= 1 + \tfrac{1}{6}\Omega^2 - \tfrac{1}{120}\Omega^4 + \tfrac{1}{120\times 42}\Omega^6 \ldots \\
&\quad + \left(\tfrac{1}{6}\Omega^2 - \tfrac{1}{120}\Omega^4 \ldots\right)^2 \\
&\quad + \left(\tfrac{1}{6}\Omega^2 \ldots\right)^3 \ldots
\end{aligned}$$

$$120\times 42 = 7\times 720 = 5040$$

$$\begin{aligned}
&= 1 + \tfrac{1}{6}\Omega^2 - \tfrac{1}{120}\Omega^4 + \tfrac{3}{15120}\Omega^6 \ldots \\
&\qquad + \tfrac{1}{36}\Omega^4 - \tfrac{1}{360}\Omega^6 \ldots \\
&\qquad\qquad + \tfrac{1}{216}\Omega^6 \ldots
\end{aligned}$$

$$15120 = 21\times 720 = 42\times 360 = 420\times 36 = 70\times 216$$

$$= 1 + \tfrac{1}{6}\Omega^2 + \left[\frac{10-3}{360}\right]\Omega^4 + \left[\frac{3-42+70}{15120}\right]\Omega^6 \ldots$$

$$= 1 + \tfrac{1}{6}\Omega^2 + \frac{7}{360}\Omega^4 + \frac{31}{15120}\Omega^6 \ldots$$

$$\text{Similarly } \alpha = \frac{-\Omega\left[1 - \frac{\Omega^2}{2} + \frac{\Omega^4}{24} - \frac{\Omega^6}{720} \ldots\right]}{\sin\Omega}$$

from above

$$= -\left[1 - \tfrac{1}{2}\Omega^2 + \tfrac{1}{24}\Omega^4 - \tfrac{1}{720}\Omega^6 \ldots\right]\left[1 + \tfrac{1}{6}\Omega^2 + \tfrac{7}{360}\Omega^4 + \tfrac{31}{15120}\Omega^6 \ldots\right]$$

$$\begin{aligned}
= -\Big[\,1 &+ \tfrac{1}{6}\Omega^2 + \tfrac{7}{360}\Omega^4 + \tfrac{31}{15120}\Omega^6 \ldots \\
&- \tfrac{1}{2}\Omega^2 - \tfrac{1}{12}\Omega^4 - \tfrac{7}{720}\Omega^6 \ldots \\
&\qquad + \tfrac{1}{24}\Omega^4 + \tfrac{1}{144}\Omega^6 \ldots \\
&\qquad\qquad - \tfrac{1}{720}\Omega^6 \ldots \Big]
\end{aligned}$$

$$\text{viz}\quad \alpha = -1 + \frac{1}{3}\Omega^2 + \left[\frac{30-15-7}{360}\right]\Omega^4$$

$$- \left[\frac{31-147+105-21}{15120}\right]\Omega^6$$

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$$= -1 + \frac{1}{3}\Omega^2 + \frac{1}{45}\Omega^4 + \frac{\cancel{34}\ 32}{42\times 360}\Omega^6$$

$$\frac{32}{2\times 21\times 4\times 2\times 45} \rightarrow \frac{2}{21\times 45} \rightarrow \frac{2}{945}$$

$$= -1 + \frac{1}{3}\Omega^2 + \frac{1}{45}\Omega^4 + \frac{2}{945}\Omega^6 \ldots$$

Ω very small, thus $\alpha + 1 \; (\sim \Omega^2)$

$$\Gamma_0 \sim 1 + a(\alpha+1) + b(\alpha+1)^2 + c(\alpha+1)^3 + \ldots$$

$$\text{if}\quad 1 + \frac{1}{6}\Omega^2 + \frac{7}{360}\Omega^4 + \frac{31}{15120}\Omega^6 \ldots$$

$$= 1 + a\left\{\frac{1}{3}\Omega^2 + \frac{1}{45}\Omega^4 + \frac{2}{945}\Omega^6 \ldots\right\}$$

$$+ b\left\{\frac{1}{9}\Omega^4 + \frac{2}{3\cdot 45}\Omega^6 \ldots\right\}$$

$$+ \frac{c}{27}\Omega^6 \ldots$$

$$\text{if}\quad a = \frac{1}{2}, \quad \frac{1}{9}b + \frac{1}{45}a = \frac{7}{360} \Rightarrow \cancel{b = \frac{7}{40} - \frac{1}{5} = -\frac{1}{40}}$$

$$\Rightarrow b = 9\left[\frac{7}{360} - \frac{1}{90}\right] = \frac{9\times 3}{360} = \frac{9}{120} = \frac{3}{40}$$

$$\text{\&}\quad \frac{c}{27} + \frac{2b}{3\cdot 45} + \frac{2a}{945} = \frac{31}{15120}$$

$$\text{i.e}\quad \frac{c}{27} = \frac{31}{42\times 360} - \frac{2\cdot 3}{40\cdot 3\cdot 45} - \frac{1}{21\times 45}$$

$$= \frac{1}{45}\left[\frac{31}{8\times 42} - \frac{1}{20} - \frac{1}{21}\right] \qquad = \frac{1}{45}\left[\frac{31}{16\times 21} - \frac{1}{20} - \frac{1}{21}\right]$$

$$= \frac{1}{45}\left[\frac{15}{16\times 21} - \frac{1}{20}\right]$$

$$= \frac{1}{180}\left[\frac{15}{4\times 21} - \frac{1}{5}\right]$$

$$= \frac{1}{180}\left[\frac{75-84}{20\times 21}\right]$$

$$= -\frac{1}{20\times 20\times 21} \ldots$$

so $c = \dfrac{-27}{20\times 20\times 21} = \dfrac{-9}{20\times 20\times 7} = \dfrac{-9}{2800}$

& $r_0 = 1 + \frac{1}{2}(\alpha+1) + \frac{3}{40}(\alpha+1)^2 - \frac{9}{2800}(\alpha+1)^3 + \dots$

For plots, see separate file, question4_5.pdf.

If we take $r = \dfrac{1 + b(\alpha+1) + c(\alpha+1)^2}{1 + c(\alpha+1)}$

$$= \frac{1 + ba + ca^2}{1 + ca} \qquad a = \alpha + 1$$

& expand, then

$$r = (1 + ba + ca^2)(1 - ca + c^2a^2 \dots)$$
$$= 1 + (b - c)a + (c - bc + c^2)a^2 + \dots$$

& if we choose to correspond to the Taylor series, we would have

$b - c = \frac{1}{2}$

$c[1 - (b - c)] = \frac{3}{40}$ so $c = \frac{3}{40} \cdot 2 = \frac{3}{20} = 0.15$

& $b = 0.65$

The choice of rational function is made so that $r \sim \alpha$ as $\alpha \to \infty$

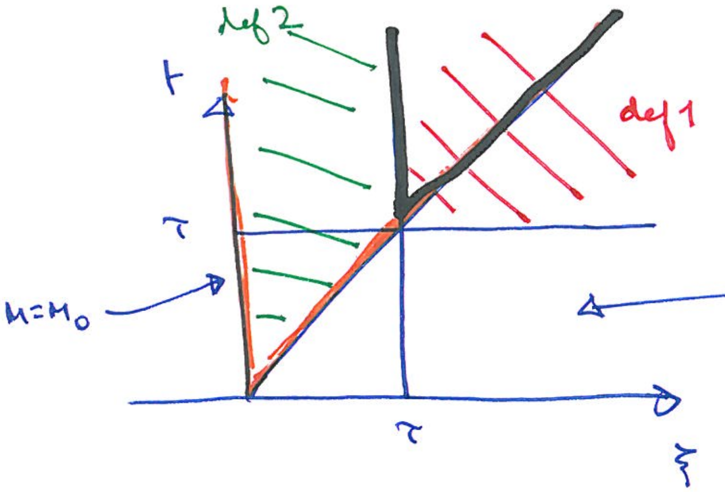
To extend accuracy we could try higher order polynomials in numerator and denominator (though it hardly seems necessary). These are Padé approximants (see Bender + Orszag ch8).

6. $M_t + M_\xi = -RM + Q$

$$Q = 2e^{-\gamma\tau} R(t-\tau, \xi-\tau) M(t-\tau, \xi-\tau) \qquad \xi > \tau, \ t > \tau \qquad (1)$$

$$= 2e^{-(\gamma_0+V_0)\tau} e^{(\gamma_0+V_0-\gamma)\xi} V_0 R_0 (t-\tau) N_0 (t-\tau), \qquad (2)$$

$$\xi < \tau, \ t > \xi$$



The definition of Q will depend on the initial ($t=0$) condition but is washed out of the system.

We focus on the region $t > \xi$.

Using $R = (1+\lambda)R_0$, $\lambda = 2e^{-\gamma\tau} - 1$, $\gamma_0 + V_0 = \gamma$, $N_0 V_0 = M_0$ we have

$$Q = \begin{cases} (1+\lambda) R M_{\tau,\tau} & (1) \\ R M_0 & (2) \end{cases}$$

and $M = M_0$ on $\xi = 0$, $t > 0$.

In $t > \xi$, $\xi < \tau$ (def 2) we have

$$M_t + M_\xi = -RM + RM_0, \quad M = M_0 \text{ on } \xi = 0$$

& the solution is obviously $M \equiv M_0$

Therefore to solve the model in $t > \xi$, $t > \tau$ (outlined in black)

we have the characteristic equation

$$\dot{M} = -RM + (1+\lambda) R M_{\tau,\tau} \qquad \& \begin{cases} M = M_0 \\ t = s > \tau \\ \xi = \tau \end{cases}$$

$$\dot{\xi} = 1$$

Thus $\xi = t - s + \tau$.

Since the initial condition is independent of t, it is clear that the solution is a function of ξ only

Defining $\eta = \dfrac{\xi - \tau}{\tau}$, $\underset{M(t,\xi)}{\overset{\shortparallel}{}} M = M_0 u(\eta)$

we have $\eta = \dfrac{t-s}{\tau}$ and thus $\dot{M} = \dfrac{M_0}{\tau} u' = -RM_0 u + (1+\lambda) R u(\eta - 1)$,

since $M_{\tau,\tau} = M(t-\tau, \xi-\tau) = M_0 u\left[\dfrac{\xi-\tau-\tau}{\tau}\right] = M_0 u(\eta-1)$

i.e. $\boxed{u' = -\alpha u - \Gamma u_1}$

where $\alpha = R\tau$, $\Gamma = -(1+\lambda)R\tau$

Note that applies for $\eta \geq 0$ $(\xi > \tau)$, and the initial function is $u = 1$ for $\eta \in [-1, 0)$

Next, define the Laplace transform

$$U(p) = \int_0^\infty u(\eta) e^{-p\eta} \, d\eta$$

Note
$$\hat{u}_1 = \int_0^\infty u(\eta-1) e^{-p\eta} \, d\eta$$

$[\eta = 1+s]$
$$= \int_{-1}^\infty u(s) e^{-p} e^{-ps} \, ds$$

$$= e^{-p} U + e^{-p} \int_{-1}^0 e^{-ps} \, ds$$

$$= e^{-p} U + \frac{e^{-p}}{p}\left[e^{p} - 1\right] = e^{-p} U + \frac{(1 - e^{-p})}{p}$$

So we have from $u' = -\alpha u - \Gamma u_1$, $u=1$, $\eta<0$

$$pU - 1 = -\alpha U - \Gamma\left[e^{-p}U + \frac{(1-e^{-p})}{p}\right]$$

or $$U = \frac{1 - \Gamma\left(\frac{1-e^{-p}}{p}\right)}{p + \alpha + \Gamma e^{-p}}$$

Hence $$U = \frac{p - \Gamma + \Gamma e^{-p}}{p(p+\alpha+\Gamma e^{-p})} = \frac{p+\alpha+\Gamma e^{-p} - (\alpha+\Gamma)}{p(p+\alpha+\Gamma e^{-p})}$$

$$= \frac{1}{p} - \frac{\alpha+\Gamma}{p(p+\alpha+\Gamma e^{-p})}$$

$$= \frac{1}{p} + \frac{(-\alpha-\Gamma)e^{p}}{p[(p+\alpha)e^{p}+\Gamma]} = \frac{1}{p} + \frac{\Lambda e^{p}}{p[(p+\alpha)e^{p}+\Gamma]}$$

where $\Lambda = -\alpha-\Gamma = -R\tau + (1+\lambda)R\tau = \lambda R\tau$

To invert this, we have $u = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} U(p)\,e^{p\eta}\,dp$

with all singularities of U in $\operatorname{Re} p < \gamma$.

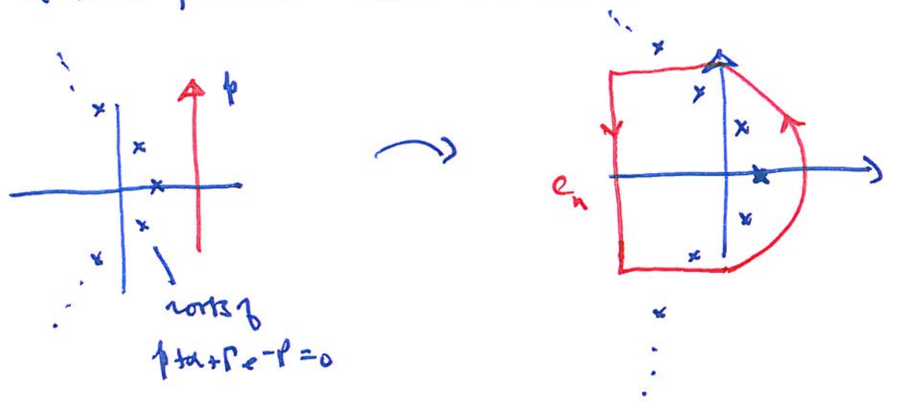
The singularities of U are at $p = 0$ (removeable, so does not contribute)

& where $p = -\alpha - \Gamma e^{-p}$

— but we know all about this! (see question 5)

For $-\Gamma > \alpha$ as here, there is one +ve root & ∞ complex roots, with a finite number to the right of the imaginary axis.

So the inversion contour can be taken as the limit of C_n as $n \to \infty$, where C_n is as shown



where the left hand part is the left side of a square of half-length $(n+\frac{1}{2})\pi$. The integral round this part vanishes as $n \to \infty$ by a version of Jordan's lemma, provided ~~that~~ $p+\alpha+\Gamma e^{-p}$ is bounded away from zero on top + bottom (clearly it goes to ∞ on the left side).

But for $p = r + i(n+\frac{1}{2})\pi$, $p+\alpha+\Gamma e^{-p} = r+\alpha + i\left[(n+\frac{1}{2})\pi + |\Gamma| e^{-r}\right]$ if n is even & $\Gamma < 0$, so $|p+\alpha+\Gamma e^{-p}| > n\pi \to \infty$ in fact.

Therefore

$$u(\eta) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{C_n} U(p)\, e^{p\eta}\, dp$$

$$= \sum_j c_j\, e^{p_j \eta} ,$$

where c_j are the residues of U at the roots

of $p + \alpha + \Gamma e^{-p} = 0$

Since $U = \dfrac{\Lambda}{p\left[p+\alpha+\Gamma e^{-p}\right]} + \dfrac{1}{p}$

& $(p+\alpha+\Gamma e^{-p})' = 1 - \Gamma e^{-p}$, these are

$$\frac{1}{p_j}\cdot\frac{\Lambda}{1-\Gamma e^{-p_j}}$$

[redacted] $= \dfrac{1}{p_j}\,\dfrac{\Lambda}{(1+p_j+\alpha)}$

[redacted]

[Note : the roots $p+\alpha+\Gamma e^{-p} = 0$

satisfy $p_n = \ln\left(\dfrac{p_n+\alpha}{\Gamma}\right) = \ln p_n - \ln\Gamma + \ln\left(1+\dfrac{\alpha}{p_n}\right) + 2n i\pi$

or $p_n = 2ni\pi\left[1 + \dfrac{\{\ln p_n - \ln\Gamma + \frac{\alpha}{p_n}\cdots\}}{2ni\pi}\right]$

& $\ln p_n = \ln 2ni\pi + \dfrac{\ln p_n}{2ni\pi}\ \cdots$

$= (\ln 2ni\pi)(1+O(\tfrac{1}{n}))$

thus $p_n \simeq 2ni\pi\left[1 + \dfrac{\ln 2ni\pi}{2ni\pi}\cdots\right] = 2ni\pi + \ln 2ni\pi + \cdots$

& $p_j \sim O(j)$ as $j\to\infty$.]