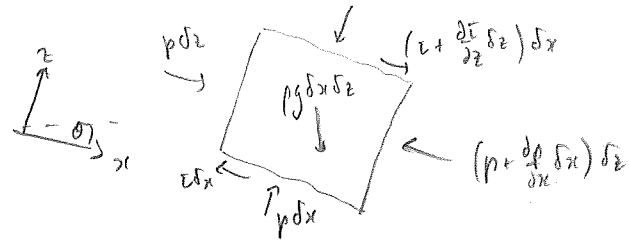


1. Sliding Glaciers

Lubrication theory (ignore longitudinal stress gradients and vertical shear stress)  $(p + \frac{\partial p}{\partial z} \delta z) \delta x$

(1) 
$$-\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z} + \rho g \sin \theta = 0$$

(2) 
$$-\frac{\partial p}{\partial z} - \rho g \cos \theta = 0$$



Fuller version of Stokes eqn.

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \sin \theta$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \cos \theta$$

Glenn's flow law (3)  $\frac{1}{2} \frac{\partial u}{\partial z} = A \tau^n$  Sliding law  $u = C \tau^m$  at  $z = b$

Surface condition: zero stress  $\Rightarrow p = \tau = 0$  at  $z = s$ .

From (2),  $p = \rho g \cos \theta (s - z)$

Then (1),  $\tau = \rho g \sin \theta (s - z)$

Then (3),  $u = C (\rho g)^m \left( \sin \theta - \cos \theta \frac{\partial s}{\partial x} \right)^m H^m + \frac{2A}{n+1} (\rho g)^n \left( \sin \theta - \cos \theta \frac{\partial s}{\partial x} \right)^n \left\{ (s-b)^{n+1} - (s-z)^{n+1} \right\}$

and then  $q = \int_b^s u dz = C (\rho g)^m \left( \sin \theta - \cos \theta \frac{\partial s}{\partial x} \right)^m H^{m+1} + \frac{2A}{(n+2)} (\rho g)^n \left( \sin \theta - \cos \theta \frac{\partial s}{\partial x} \right)^n H^{n+2}$   $H = s - b$

$\int_b^s (s-b)^{n+1} - (s-z)^{n+1} dz = (s-b)^{n+2} - \frac{1}{n+2} (s-b)^{n+2} = \frac{n+1}{n+2} (s-b)^{n+2}$

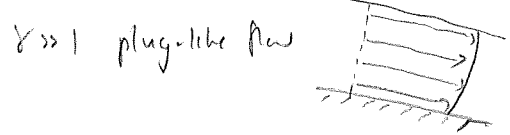
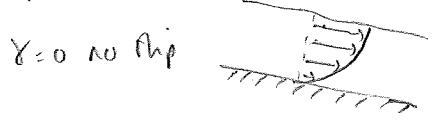
Mass conservation  $\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a$  (from first principles, or derive from continuity equations  $u_x + w_z = 0$  with  $w = u \frac{\partial s}{\partial x}$  at  $z = b$ ,  $w = u \frac{\partial s}{\partial x} + \frac{\partial s}{\partial t} - a$  at  $z = s$ )

Non-dimensionalise by choosing  $[H]$  so that  $2A (\rho g \sin \theta)^n [H]^{n+2} = [q] = [a][x]$ , and  $[t] = \frac{[H][x]}{[q]}$

Then  $\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} \left[ (1 - \mu H_x) \frac{H^{n+2}}{n+2} + \gamma (1 - \mu H_x) \frac{H^{m+1}}{m+1} \right] = a$  where  $\gamma = \frac{C (\rho g \sin \theta)^{m-n}}{2A} (m+1) [H]^{m-n}$

$\mu = \frac{[H]}{[x] \sin \theta}$

$\gamma$  represents the importance of sliding:



$\mu=0$ .  $a=1-x$ . Equation is hyperbolic, first order.

$$H_t + \left[ \frac{H^{n+2}}{n+2} + \gamma \frac{H^{m+1}}{m+1} \right]_x = a$$

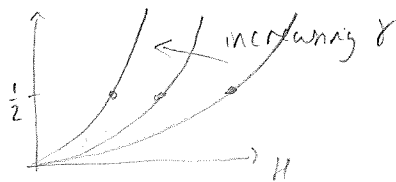
$\Rightarrow$  boundary conditions  $H=0$  at  $x=0$ .

(also  $H=0$  at  $x=x_s$  determines the location  $x_s$  of the end of the glacier).

Steady state given by  $\frac{H^{n+2}}{n+2} + \gamma \frac{H^{m+1}}{m+1} = \int_0^x a dx = x(1 - \frac{1}{2}x)$

The length of the glacier is always  $\leq 2$ . Since the LHS of above expression is an increasing function of  $H$ , the maximum depth occurs when the RHS is maximum, i.e. at  $x=1$ , where

$$\frac{H^{n+2}}{n+2} + \gamma \frac{H^{m+1}}{m+1} = \frac{1}{2}$$



Clearly from the graph, increasing  $\gamma$  reduces the solution for  $H$ . For large  $\gamma$ ,  $H \sim \left( \frac{m+1}{4\gamma} \right)^{\frac{1}{m+1}}$

## 2. Second evolution of glacier.

$$H_t + H^{\alpha+1} H_x = a. \quad H=0 \text{ at } x=0.$$

$$a = a_0(x) + a_1 \sin \omega t$$

Write  $H = H_0(x) + H_1(x,t)$  (and suppose  $H_1 \ll H_0$ , so we may linearise the characteristics and write  $\frac{dx}{dt} \approx H_0^{\alpha+1}$ ). Then we are solving

$$H_{1,t} + H_0^{\alpha+1} H_{0,x} + H_0^{\alpha+1} H_{1,x} = a_0 + a_1 \sin \omega t.$$

Taking the temporal mean, we have  $H_0^{\alpha+1} H_{0,x} = a_0$ ,  $H_0 = 0$  at  $x=0$ .

$$\Rightarrow H_0 = \left[ (\alpha+1) \int_0^x a_0(\hat{x}) d\hat{x} \right]^{\frac{1}{\alpha+2}}$$

To solve for  $H_1$ , make change of variable  $H_0(x)^{\alpha+1} \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \xi}$  by defining

$$\xi = \int_0^x \frac{1}{H_0^{\alpha+1}(\hat{x})} d\hat{x}.$$

Then  $H_{1,t} + H_{1,\xi} = a_1 \sin \omega t$ , with  $H_1 = 0$  at  $x=0$ .

characteristic  $\dot{\xi} = 1$ ,  $\dot{H}_1 = a_1 \sin \omega t$  with  $\xi = 0$ ,  $H_1 = 0$  at  $t = t_0$ .

$$\Rightarrow \xi = t - t_0, \quad H_1 = \frac{a_1}{\omega} (\cos \omega t_0 - \cos \omega t)$$

$$\Rightarrow H_1 = \frac{a_1}{\omega} [\cos \omega(t - \xi) - \cos \omega t]$$

$$[\cos A - \cos B = 2 \sin \left( \frac{A+B}{2} \right) \sin \left( \frac{B-A}{2} \right)]$$

$$= \frac{2a_1}{\omega} \sin \omega \left( t - \frac{\xi}{2} \right) \sin \frac{\omega \xi}{2}$$

$$\text{So } H = H_0(x) + \frac{2a_1}{\omega} \sin \omega \left( t - \frac{\xi}{2} \right) \sin \frac{\omega \xi}{2}$$

Note the change of variable  $x \rightarrow \xi$  is not necessary, we could just solve  $H_{1,t} + H_0^{\alpha+1} H_{1,x} = a_1 \sin \omega t$  by characteristics  $\dot{x} = H_0(x)^{\alpha+1}$  with  $x=0$  at  $t=t_0 \Rightarrow \int_0^x \frac{dx}{H_0(x)^{\alpha+1}} = t - t_0$ .

The perturbations propagate as waves moving down the glacier (at speed 2 in distance space variable).

The amplitude of perturbations decreases with increasing  $\omega$  - 10 century scale changes have a large effect than seasonal variations (since  $\omega$  is smaller).

The assumption  $H_1 \ll H_0$  is valid provided  $a_1/\omega \ll 1$ , i.e. provided  $\omega$  sufficiently large.

For  $\omega \ll 1$ , the variation of accumulation are slow compared to the evolution of the glacier

so a quasi-steady approximation  $\frac{H^{\alpha+2}}{\alpha+2} = \int_0^x a_0(\hat{x}) + a_1 \sin \omega t dx$  is appropriate.

### 3. Ice sheet

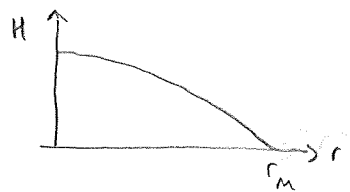
Vertical momentum balance (2)  $0 = -\frac{\partial p}{\partial z} - \rho g$

Radial " (3)  $0 = -\frac{\partial p}{\partial r} + \frac{\partial \tau}{\partial z}$  ( $\bar{\tau} = \tau_{rz}$ )

Depth-integrated mass conservation (1)  $\frac{\partial H}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r q) = a$ , where  $q = \int_0^H u dz$

and where  $u(r, z)$  is the radial velocity.

Flow law (4)  $\frac{\partial u}{\partial r} = 2A \tau^\alpha$



From (2) with  $p=0$  at  $z=H$ ,  $p = \rho g(H-z)$ .

(3)  $\tau=0$  at  $z=H$ ,  $\tau = -\rho g \frac{\partial H}{\partial r} (H-z)$

( $\frac{\partial H}{\partial r} < 0$  by assumption)

(4)  $u=0$  at  $z=0$   $u = 2A \left( -\rho g \frac{\partial H}{\partial r} \right)^\alpha \int_0^z (H-\hat{z})^\alpha d\hat{z}$ .

$$= \frac{2A \left( -\rho g \frac{\partial H}{\partial r} \right)^\alpha}{\alpha+1} \left[ H^{\alpha+1} - (H-z)^{\alpha+1} \right]$$

Then  $q = \frac{2A \left( -\rho g \frac{\partial H}{\partial r} \right)^\alpha}{\alpha+1} \left[ H^{\alpha+1} H - \frac{H^{\alpha+2}}{\alpha+2} \right] = \frac{2A (\rho g)^\alpha}{\alpha+2} H^{\alpha+2} \left( -\frac{\partial H}{\partial r} \right)^\alpha$

so (1) becomes  $\frac{\partial H}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{2A (\rho g)^\alpha}{\alpha+2} r H^{\alpha+2} \left( -\frac{\partial H}{\partial r} \right)^{\alpha-1} \frac{\partial H}{\partial r} \right] + a$

Scale  $r \sim [r]$ ,  $a \sim [a]$ ,  $H \sim [H]$ ,  $t \sim [t]$ , with  $[t] = \frac{[H]}{[a]}$ , and  $\frac{2A (\rho g)^\alpha}{\alpha+2} \frac{[H]^{2\alpha+2}}{[r]^{\alpha+1}} = [a]$

Then dimensionless equation is  $\frac{\partial H}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r H^{\alpha+2} \left( -\frac{\partial H}{\partial r} \right)^{\alpha-1} \frac{\partial H}{\partial r} \right] + a$

Given  $a(r)$ , a steady-state ice sheet has  $q = H^{\alpha+2} \left( -\frac{\partial H}{\partial r} \right)^\alpha = \frac{1}{r} \int_0^r \hat{r} a(\hat{r}) d\hat{r}$  (radial ice flux)

and therefore  $H^{\frac{\alpha+2}{\alpha}} \frac{\partial H}{\partial r} = - \left( \frac{1}{r} \int_0^r \hat{r} a(\hat{r}) d\hat{r} \right)^{\frac{1}{\alpha}}$ , with  $H=0$  at  $r=r_m$ .

$$\Rightarrow \frac{1}{2\alpha+2} H^{\frac{2\alpha+2}{\alpha}} = \int_r^{r_m} \left[ \frac{1}{\hat{r}} \int_0^{\hat{r}} \hat{r} a(\hat{r}) d\hat{r} \right]^{\frac{1}{\alpha}} d\hat{r}$$

$$\Rightarrow H(r) = \left\{ \frac{2(\alpha+1)}{\alpha} \int_r^{r_m} \left[ \frac{1}{\hat{r}} \int_0^{\hat{r}} \hat{r} a(\hat{r}) d\hat{r} \right]^{\frac{1}{\alpha}} d\hat{r} \right\}^{\frac{\alpha}{2(\alpha+1)}}$$

Ice flux zero at  $r_m$  for a land-terminating ice cap, so  $r_m$  is such that

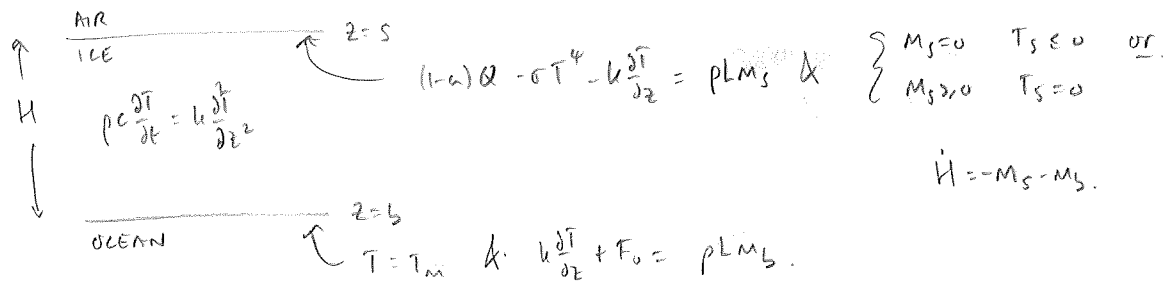
$$\int_0^{r_m} \hat{r} a(\hat{r}) d\hat{r} = 0$$

If  $a(r) = 1-r$ , we have  $\frac{1}{r} \int_0^r \hat{r} a(\hat{r}) d\hat{r} = \frac{1}{r} \int_0^r \hat{r}(1-\hat{r}) d\hat{r} = \frac{1}{r} \left( \frac{1}{2} r^2 - \frac{1}{3} r^3 \right) = \frac{1}{3} r \left( \frac{3}{2} - r \right)$

so  $r_m = \frac{3}{2}$ , and  $H = \left\{ \frac{2(\lambda+1)}{\lambda} \int_r^{3/2} \left[ \frac{1}{3} \hat{r} \left( \frac{3}{2} - \hat{r} \right) \right]^\lambda d\hat{r} \right\}^{1/(\lambda+1)}$

In the limit  $\lambda \rightarrow \infty$ ,  $H$  becomes  $H = \left\{ 2 \int_r^{3/2} d\hat{r} \right\}^{1/2} = (3-2r)^{1/2}$

4. Sea Ice



For the surface energy balance ( $z=s$ ), the left hand side is the net heat flux into the surface (shortwave radiation - longwave outgoing radiation + conductive flux from ice), which must be zero (i.e. continuity of heat flux) if below melting point, and must balance the latent heat of melting if at the melting point.

For the basal boundary, the temperature must be at the melting point, and the net heat flux into the boundary balances the melting rate.

Non-dimensionalise by writing  $T = T_m + [T] \hat{T}$ ,  $z = [H] \hat{z}$ ,  $H = [H] \hat{H}$ ,  $t = [t] \hat{t}$ ,  $m = \frac{[H]}{[t]} \hat{m}$

Surface energy balance becomes  $(1-\alpha)Q - \sigma T_m^4 \left(1 + \frac{[T]}{T_m} \hat{T}\right)^4 - \frac{k [T]}{[H]} \frac{\partial \hat{T}}{\partial \hat{z}} = \rho L [H] \hat{m}_s$

$1 + 4 \frac{[T]}{T_m} \hat{T} + \dots$

So choose scales for  $[H] = \frac{k}{4\sigma T_m^3}$  and  $[t] = \frac{\rho L [H]^2}{k [T]}$

Then surface energy balance becomes  $\frac{(1-\alpha)Q - \sigma T_m^4}{4\sigma T_m^3 [T]} - \hat{T} - \frac{\partial \hat{T}}{\partial \hat{z}} = \hat{m}_s$

Basal condition becomes

$\hat{T} = 0 \quad \& \quad \frac{\partial \hat{T}}{\partial \hat{z}} + \frac{[H] F_0}{k [T]} = \hat{m}_b$ , and  $\hat{H} = -\hat{m}_b - \hat{m}_s$

And the heat equation becomes

$\frac{1}{S} \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial^2 \hat{T}}{\partial \hat{z}^2}$  where  $S = \frac{k [t]}{\rho c [H]^2} = \frac{L}{c [T]}$

Given parameter values:  $k \approx 2 \text{ W m}^{-1} \text{ K}^{-1}$ ,  $\sigma = 5.67 \times 10^{-8} \text{ W K}^{-2}$ ,  $T_m = 273 \text{ K}$ , we have  $[H] \approx 0.43 \text{ m}$  and with  $\rho \approx 10^3 \text{ kg m}^{-3}$ ,  $L = 3.3 \times 10^5 \text{ J kg}^{-1}$  and choosing  $[T] = 10 \text{ K}$ , (so  $\frac{k [T]}{[H]} \approx 46 \text{ W m}^{-2}$ ) this gives  $[t] = 2.8 \times 10^6 \text{ s} \approx 1 \text{ month}$ . With  $c \approx 2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$ , this gives  $S \approx 17$ .

If  $S \gg 1$ , the heat equation is quasi-steady,  $\frac{\partial^2 T}{\partial z^2} = 0 \Rightarrow \boxed{T = T_s \left( \frac{z-b}{H} \right)} \Rightarrow \frac{\partial T}{\partial z} = \frac{T_s}{H}$

(writing  $T_s$  for temp at the surface  $z=b$ , and using  $T=0$  at  $z=b$ )

The surface energy balance then gives  $\hat{Q} - T_s - \frac{T_s}{H} = M_s$ , with  $\begin{cases} M_s = 0 & T_s \leq 0 \\ \text{or } M_s > 0 & T_s = 0 \end{cases}$

Hence,  $\boxed{\begin{matrix} \text{if } \hat{Q} \leq 0, & T_s = \frac{\hat{Q}H}{(1+H)} & \text{and } M_s = 0 \\ \hat{Q} > 0 & T_s = 0 & \text{and } M_s = \hat{Q} \end{matrix}}$

The basal energy balance gives  $M_b = \hat{F}_0 + \frac{T_s}{H} = \begin{cases} \hat{F}_0 + \frac{\hat{Q}}{1+H} & \hat{Q} \leq 0 \\ \hat{F}_0 & \hat{Q} > 0 \end{cases}$

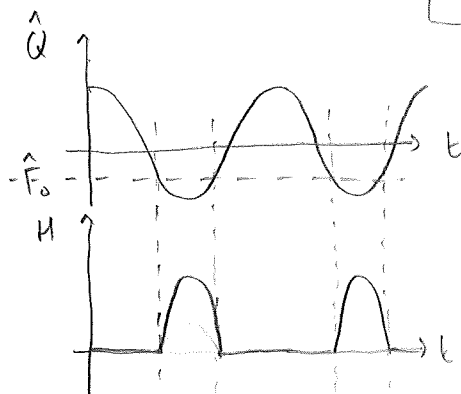
Hence  $\dot{H} = -M_b - M_s = \begin{cases} -\hat{F}_0 - \frac{\hat{Q}}{1+H} & \hat{Q} \leq 0 \\ -\hat{F}_0 - \hat{Q} & \hat{Q} > 0 \end{cases}$  (all basal melting / freezing)  
(both surface / basal melting)

But this assumes that there is some ice! i.e. that  $H > 0$ . If  $H = 0$ , ice can only start to grow if  $\dot{H} > 0$ , and this occurs in the first of the two cases above provided  $\hat{Q} < -\hat{F}_0$ . Otherwise  $\dot{H} = 0$  when  $H = 0$ .

If  $\hat{Q} = \hat{Q}_0 + \cos \omega t$ , with  $\omega \ll 1$ , the evolution of  $H$  is quasi-steady (rescale to  $\frac{1}{\omega}$  to see this formally). The only non-zero steady state is  $H = -1 - \frac{\hat{Q}}{\hat{F}_0}$ , which requires  $\hat{Q} < -\hat{F}_0$ .

Otherwise  $H = 0$ . So

$$H = \begin{cases} -1 - \frac{\hat{Q}_0}{\hat{F}_0} - \frac{\cos \omega t}{\hat{F}_0} & \text{when } \cos \omega t < -(\hat{F}_0 + \hat{Q}_0) \\ 0 & \text{otherwise} \end{cases}$$



The ice-covered fraction of the year is given by

$$\frac{1}{\pi} \cos^{-1} \left( \frac{\hat{F}_0 + \hat{Q}_0}{\hat{F}_0} \right)$$

