Problem sheet 1

1. **Lapse Rates** Suppose the atmosphere is dry, adiabatic and hydrostatic, and obeys the ideal gas law, so that

$$\rho c_p \frac{\mathrm{d}T}{\mathrm{d}z} - \frac{\mathrm{d}p}{\mathrm{d}z} = 0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g, \qquad p = \frac{\rho R T}{M_a},$$

with $T = T_s$ and $p = p_s$ at $z = 0$. The gravitational acceleration g, specific heat capacity c_p, molecular weight M_a, and gas constant R are all constants.

Find T, p and ρ as functions of z, and confirm that $p/p_s = (T/T_s)^{M_a c_p/R}$. Explain why an appropriate definition for the depth of this atmosphere is $d = c_p T_s/g$. Roughly how much thinner is the air at the top of Mt Everest than at sea level according to this model?

Parameter values: $c_p = 10^3$ J kg^{-1} K^{-1}, $M_a = 29 \times 10^{-3}$ kg mol^{-1}, $R = 8.3$ J K^{-1} mol^{-1}, and $g = 9.8$ m s^{-2}.

2. **Two stream approximation** The radiative transfer equation for a one-dimensional atmosphere is

$$\cos\theta \frac{\partial I}{\partial z} = -\kappa\rho(I - B),$$

where $I(z,\theta)$ is the intensity of longwave radiation, $B(z) = \sigma T^4/\pi$, $T(z)$ is the air temperature, $\rho(z)$ is the density, and the adsorption coefficient κ can be considered constant.

(i) Derive the two-stream approximation,

$$\frac{1}{2}\frac{\mathrm{d}I_+}{\mathrm{d}z} = -\kappa\rho(I_+ - B), \qquad -\frac{1}{2}\frac{\mathrm{d}I_-}{\mathrm{d}z} = -\kappa\rho(I_- - B),$$

where $I_\pm$ are average intensities over upward and downward directions, and the upward and downward energy fluxes are $F_\pm = \pi I_\pm$. Write down appropriate boundary conditions for $I_\pm$ if there is no incoming radiation from the top of the atmosphere $z = d$, and the surface temperature at $z = 0$ is T_s (use the Stefan Boltzmann law). Give an appropriate definition of the effective longwave emission temperature T_e.

(ii) Now make the assumption of local radiative equilibrium, and suppose $\rho(z)$ is known. Show that the net upwards flux $F = F_+ - F_-$ is constant, and solve for $I_\pm$ in terms of T_s and the optical depth $\tau = \int_z^d \kappa\rho \, \mathrm{d}z$.

Use your solution to find the greenhouse factor $\gamma = T_e^4/T_s^4$, and to sketch the air temperature as a function of height.

(iii) Suppose instead that $T(z)$ is known, and is not necessarily determined by local radiative equilibrium. Solve for I_+ and hence show that the greenhouse factor is given by

$$\gamma = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\tau}\,\mathrm{d}\tau, \qquad (\star)$$

where τ and τ_s are defined as above.

(iv) For the atmosphere in question 1, show that $T/T_s = (\tau/\tau_s)^{R/M_a c_p}$ and $\tau_s = \kappa p_s/g$, and hence give an expression for $\gamma(\tau_s)$ in this case. Show that it can be approximated by

$$\gamma \sim 1 - \frac{8R}{4R + M_a c_p}\tau_s \qquad \text{and} \qquad \gamma \sim (2\tau_s)^{-4R/M_a c_p}\,\Gamma\left(1 + \frac{4R}{M_a c_p}\right)$$

in the limits $\tau_s \ll 1$ and $\tau_s \gg 1$, respectively, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ is the Gamma function.

[*If keen, evaluate the expression for $\gamma(\tau_s)$ numerically and check these approximations.*]