Problem sheet 2

1. **Runaway greenhouse effect** Show that for T close to a reference temperature T_0, the solution of the Clausius-Clapeyron equation for saturation vapour pressure p_{sv} as a function of temperature T is approximately

$$p_{sv} \approx p_{sv0} \exp\left[a\left(\frac{T-T_0}{T_0}\right)\right].$$

where $a = M_v L/RT_0$, and we may take $T_0 = 273$ K at $p_{sv0} = 600$ Pa (the *triple point*, where ice, water, and vapour can all exist at equilibrium).

If the longwave radiation from a planet is $\sigma\gamma T^4$, the solar flux is Q, the planetary albedo is zero, and the greenhouse factor is given in terms of vapour pressure p by

$$\gamma^{-1/4} = 1 + b(p_v/p_{sv0})^c,$$

where b and c are constants, find the equilibrium mean surface temperature T in terms of p_v.

Hence show that the occurrence of a runaway greenhouse effect is controlled by the intersection of the two curves

$$\theta = 1 + \delta\xi, \quad \theta = \alpha(1 + be^{\xi}),$$

where $\delta = 1/ac$, $\alpha = (Q/4\sigma T_0^4)^{1/4}$. Show that runaway occurs if $\alpha > \alpha_c$, where

$$\alpha_c + \delta = 1 + \delta \ln(\delta/b\alpha_c),$$

and, if δ is small, that $\alpha_c \approx 1 + \delta \ln(\delta/b) - \delta$.

Estimate values of α and δ appropriate to the present Earth, and comment on the implications of these values for climatic evolution if we choose $b = 0.06$, $c = 0.25$. What are the implications for Venus, where the solar flux is twice as great?

Parameter values: $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, $M_v = 18 \times 10^{-3}$ kg mol^{-1}, $L = 2.5 \times 10^6$ J kg^{-1}, $R = 8.3$ J K^{-1} mol^{-1}, $Q = 1370$ W m^{-2}.

2. **Ice albedo feedback** A model for the mean temperature T of the Earth's atmosphere is

$$c\frac{dT}{dt} = R_i - R_o, \qquad R_i = \frac{1}{4}Q(1-a), \quad R_o = \sigma\gamma T^4,$$

where σ and γ are constant, and a varies piecewise linearly with temperature, such that $a = a_+$ for $T < T_i$, $a = a_-$ for $T > T_w$ (with $a_- < a_+$), and $a(T)$ is linear for $T_i \le T \le T_w$.

Show graphically that there can be multiple steady states for some range of Q provided

$$\frac{T_w - T_i}{T_i} < \frac{a_+ - a_-}{4(1-a_+)}.$$

[*Hint: consider the slopes* $R_i'(T)$ *and* $R_o'(T)$ *at* $T = T_i$.]

Show that in that case the upper and lower solutions are stable, but the intermediate one is unstable.

[*Harder*] Find the range $Q_- \le Q \le Q_+$ for which multiple steady states occur (*i.e.* give formulae for $Q_\pm$ in terms of the other parameters), taking care to distinguish the cases

$$\frac{T_w - T_i}{T_w} \gtrless \frac{a_+ - a_-}{4(1-a_-)}.$$

3. **Carbon cycles** Consider the dimensionless model from lectures for the evolution of albedo and partial pressure of atmospheric CO_2,

$$\dot{a} = f(a,p) = B(\Theta) - a,$$
$$\dot{p} = g(a,p) = \alpha(1 - wp^{\mu}e^{\Theta}),$$

where $\Theta(a,p) = \frac{q(1-a)-1}{\nu} + \lambda p$, and $B(\theta)$ is a monotonic function decreasing from a_+ to a_-. Here, μ, α, ν, λ, w. and q are all constant parameters.

Show that the p nullcline, $a = G(p)$ say, is a monotonically increasing function of p, and that the a nullcline, $a = F(p)$, is a monotonically decreasing function if $-B'(\theta) < \nu/q$ for all θ, but is multivalued if $-B'(\theta) > \nu/q$ for some range of θ.

Now suppose that the a nullcline is indeed multivalued, but that there is always a unique steady state, which may lie on the lower, intermediate, or upper branch depending on the value of w. Sketch the nullclines for each of these cases. By considering the signs of the partial derivatives of $f(a,p)$ and $g(a,p)$ (but without detailed calculation), show that steady states on the upper or lower branch are stable, but the intermediate state will be unstable if α is small enough. How would you expect the solutions to behave if $\alpha \ll 1$?

4. **Ocean carbon** A model for the global climate is

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma(p)T^4, \qquad t_i\frac{\mathrm{d}a}{\mathrm{d}t} = a_0(T) - a,$$
$$\frac{A_E M_{CO_2}}{gM_a}\frac{\mathrm{d}p}{\mathrm{d}t} = v - h(p - p_s), \qquad \rho_O V_O\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{h(p-p_s)}{M_{CO_2}} - bC,$$

where $p_s = C/K$ is the effective partial pressure of CO_2 in the ocean (*i.e.* the partial pressure of a gas in equilibrium with the water). Briefly explain the meaning of the terms in this model and the physical principles on which it is based.

Roughly estimate the timescales involved in this model using the parameter values listed below. Hence show that a suitable quasi-steady approximation of the model is

$$t_i\dot{a} = a_0(T) - a,$$
$$t_C\dot{C} = C_v - C,$$

where $t_C = \rho_O V_O/b$, $C_v = v/M_{CO_2}b$, and where

$$p \approx \frac{C}{K} + \frac{v}{h}, \qquad T \approx \left(\frac{Q(1-a)}{4\sigma\gamma(p)}\right)^{1/4}.$$

Assuming that the ocean was in equilibrium with pre-industrial emissions, infer the value of those emissions (*i.e.* v) given the present day value of $C \approx 2 \times 10^{-3}$ mol kg^{-1}, and estimate the pre-industrial value of atmospheric CO_2 given by p.

Suppose the present-day emissions $v \approx 30 \times 10^{12}$ kg y^{-1} are maintained indefinitely. Use the model to show that on a timescale of centuries p will reach an approximate equilibrium and find its value. Show that thereafter p will continue to increase, on a timescale of millennia. What is the eventual value of p?

Parameter values: $c = 10^7$ J m^{-2} K^{-1}, $Q = 1370$ W m^{-2}, $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, $t_i = 10^4$ y, $A_E = 5.1 \times 10^{14}$ m^2, $g = 9.8$ m s^{-2}, $M_{CO_2} = 44 \times 10^{-3}$ kg mol^{-1}, $M_a = 29 \times 10^{-3}$ kg mol^{-1}, $h = 0.73 \times 10^{12}$ kg y^{-1} Pa^{-1}, $\rho_O = 10^3$ kg m^{-3}, $V_O = 1.35 \times 10^{18}$ m^3, $b = 0.83 \times 10^{16}$ kg y^{-1}, $K = 7.1 \times 10^{-5}$ mol kg^{-1} Pa^{-1}.