Problem sheet 3

1. **Overland flow** Overland flow on a hill slope is described by the equation

$$\frac{\partial h}{\partial t} + ch^m \frac{\partial h}{\partial x} = E,$$

where $E = P - I$ is the excess rainfall rate, being the difference between precipitation and infiltration rates. The equation is to be solved in $x > 0$, and the initial and boundary condition are

$$h = 0 \quad \text{at} \quad t = 0, x > 0 \quad \text{and} \quad x = 0, t > 0.$$

First consider the case of constant $E > 0$. Solve the equation and sketch the solution at various times.

Next, consider the case where $E(t)$ is time dependent, such that $E \geq 0$ for $0 \leq t \leq t_*$, and $E < 0$ for $t > t_*$. Find an implicit expression for the solution in terms of integrals of E for $t \leq t_*$.

For $t > t_*$, a drying front moves down the slope (behind which $h = 0$). Determine the position of the front $x_d(t)$ as a function of time and hence find the complete solution for all $t > 0$.

2. **St Venant equations** Derive the St Venant equations from first principles in the form

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{\tau \ell}{\rho A} - g\frac{\partial \bar{h}}{\partial x},$$

indicating the meaning of the various terms.

Manning's law corresponds to taking $\tau = \rho g n^2 u^2 / R^{1/3}$, where $R = A/\ell$ is the hydraulic radius. Assuming a triangular cross-section with transverse bed angle β, find appropriate expressions for τ and $\bar{h}$ in terms of u and A.

Non-dimensionalise the resulting equations using a length scale L and discharge scale Q to obtain

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0,$$

$$\varepsilon F^2 \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A^{2/3}} - \varepsilon \frac{\partial}{\partial x}(A^{1/2}),$$

and define the parameters ε and F. Assuming that $\varepsilon \ll 1$ and $F \ll 1$, show that A satisfies the approximate equation

$$\frac{\partial A}{\partial t} + \frac{4}{3}A^{1/3}\frac{\partial A}{\partial x} = \frac{1}{4}\varepsilon \frac{\partial}{\partial x}\left(A^{5/6}\frac{\partial A}{\partial x}\right).$$

A sluice gate on the river is suddenly opened so that the cross-sectional area there increases from A_- to A_+. The hydrograph is measured a distance L downstream. Sketch the hydrograph for the cases (i) $\varepsilon = 0$ and (ii) $0 < \varepsilon \ll 1$ (no detailed calculation is required).

3. **Surface waves** Show that with a suitable choice of non-dimensionalisation, the St Venant equations for a triangular-shaped cross section with Manning's roughness law, can be written in the form

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0,$$

$$F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A^{2/3}} - \frac{1}{2A^{1/2}}\frac{\partial A}{\partial x},$$

giving the definition of F. Write down the uniform steady state with dimensionless discharge 1.

Show that small perturbations to the steady state can propagate up and downstream if $F < F_1$, but can only propagate downstream if $F > F_1$; and that they are unstable if $F > F_2$. Give the values of F_1 and F_2.

4. **Anti-dunes** A simple model of bed erosion based on the St Venant equations can be written in dimensionless form as

$$\varepsilon h_t + (hu)_x = 0,$$

$$F^2(\varepsilon u_t + uu_x) = -\eta_x + \delta\left(1 - \frac{u^2}{h}\right),$$

$$h(\varepsilon c_t + uc_x) = E(u) - c = -s_t,$$

where $h = \eta - s$, and $E(1) = 1$. Briefly explain the meaning of the terms in this model, and the physical significance of the dimensionless parameters ε, δ and F.

By considering the stability of the steady state $u = h = c = 1$, and assuming that $\varepsilon \ll 1$ while δ and F are $\mathcal{O}(1)$, show that instability can occur depending on the sign of $E'(1)$ and the size of F.

Find the phase difference between surface and bed profiles in the limit of small and large wavenumbers ($k \to 0$ and $k \to \infty$).

5. **Eddy-viscosity model** Derive the Exner equation relating bed elevation s and bedload transport q.

Supposing the bedload is a function of the shear stress $q = q(\tau)$, show that the equation can be written in dimensionless form as,

$$\frac{\partial s}{\partial t} + q'(\tau)\frac{\partial \tau}{\partial x} = 0.$$

An eddy-viscosity model for turbulent flow over linearised topography leads to the following approximate expression for the dimensionless shear stress,

$$\tau = \left[1 - s + \int_0^\infty K(\xi)\frac{\partial s}{\partial x}(x - \xi, t)\,\mathrm{d}\xi\right],$$

where the kernel is $K(x) = \mu/x^{1/3}$, and $\mu > 0$ is constant.

Making use of this expression, examine whether linear perturbations to the steady state $s = 0$ are unstable. Which direction do the perturbations travel?

[*Hint: in your calculation you will need to evaluate the integral* $\int_0^\infty \xi^{-1/3}e^{-ik\xi}\,d\xi$, *for which you can use contour integration to find the value* $e^{-i\pi/3}\Gamma(\frac{2}{3})k^{-2/3}$, *where* $\Gamma(\nu) = \int_0^\infty t^{\nu-1}e^{-t}\,dt$ *is the gamma function.*]