# Problem sheet 4

1. **Sliding glaciers** Consider a two-dimensional valley glacier of depth $H(x,t)$ flowing over a bed inclined at an angle $\theta$ to the horizontal, with net accumulation $a(x,t)$.

   Use lubrication theory (the shallow ice approximation), assuming Glen's flow law with a constant rate factor $A$, and a sliding law of the form $u_b = C\tau_b^m$, to derive an approximate model for the evolution of $H$.

   Non-dimensionalise the model to the form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\left(1-\mu\frac{\partial H}{\partial x}\right)^n \frac{H^{n+2}}{n+2} + \gamma\left(1-\mu\frac{\partial H}{\partial x}\right)^m \frac{H^{m+1}}{m+1}\right] = a,$$

   giving definitions of the parameters $\mu$ and $\gamma$ (you should assume known scales for the accumulation rate $[a]$ and distance $[x]$).

   To what do the limits $\gamma = 0$ and $\gamma \gg 1$ correspond? Sketch the velocity profile with depth in each case.

   Consider the approximation $\mu = 0$, and suppose that the dimensionless accumulation is $a = 1 - x$, where $x = 0$ is the head of the glacier. What are appropriate boundary conditions for the equation in this case? Give an expression for the steady state depth and show that the length of the glacier is independent of $\gamma$, but that its maximum depth decreases with increasing $\gamma$.

2. **Seasonal evolution of a glacier** A glacier of depth $H$ is described by the approximate dimensionless equation

$$H_t + H^{n+1}H_x = a, \qquad H = 0 \quad \text{at} \quad x = 0,$$

   where the accumulation rate function $a$ varies sinusoidally in time about a space-dependent mean,

$$a(x,t) = a_0(x) + a_1 \sin \omega t,$$

   where $a_1$ is constant.

   Assuming that $H(x,t)$ varies by only a small amount from its mean $H_0(x)$, linearise the characteristics of the model and hence determine the approximate solution for the perturbed surface $H_1(x,t) = H - H_0$.

   What can you say about the effect of century-scale changes in accumulation as compared to seasonal variations?

   What assumptions are needed about the size of $\omega$ to validate the approximation of small deviation from the mean? What alternative approximation might be used to understand the behaviour in the limit $\omega \ll 1$?

3. **Ice sheets** Write down appropriate lubrication equations to describe the flow of a radially symmetric ice sheet flowing over a flat bed, assuming no slip at the bed, and assuming Glen's flow law for the ice in the form $\dot{\varepsilon} = A\tau^n$. Show that the ice thickness satisfies a dimensionless equation of the form

$$\frac{\partial H}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rH^{n+2}\left|\frac{\partial H}{\partial r}\right|^{n-1}\frac{\partial H}{\partial r}\right) + a,$$

   where $a$ is the net accumulation.

Derive an expression for the steady-state ice thickness given an accumulation function $a(r)$, and assuming that the ice thickness is zero at the margin $r = r_m$. What determines the location of the margin $r_m$ if the ice sheet is land-terminating?

Calculate $r_m$ for the case $a(r) = 1 - r$, and show that in the limit $n \to \infty$ (a pure plasticity model for the ice), the maximum dimensionless ice thickness is $\sqrt{3}$.

[*Hint: the approximate vertical and horizontal (radial) momentum equations are the same as in two dimensions, the only significant deviatoric stress component being* $\tau = \tau_{rz}$.]

4. **Sea ice** The temperature $T(z,t)$ of a layer of sea ice $b(t) < z < s(t)$ is governed by the heat equation,

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2},$$

where $\rho$ is the ice density, $c$ is the heat capacity, and $k$ is the conductivity. Energy balance at the upper surface $z = s$ is described by

$$(1-a)Q - \sigma T^4 - k\frac{\partial T}{\partial z} = \rho L m_s \qquad \text{with} \begin{cases} \text{either} & m_s = 0, \quad T \le T_m, \\ \text{or} & m_s \ge 0, \quad T = T_m, \end{cases}$$

where $Q(t)$ is the shortwave radiative flux, $a$ is the albedo, $\sigma$ is the Stefan-Boltzmann coefficient, $L$ is the latent heat, and $m_s$ is the surface melt rate. The conditions at the lower surface $z = b$ are

$$T = T_m, \qquad k\frac{\partial T}{\partial z} + F_o = \rho L m_b,$$

where $F_o$ is the (prescribed) heat flux from the ocean below and $m_b$ is the basal melt rate. The rate of change of the ice thickness $H(t) = s - b$ is given by $\dot{H} = -m_b - m_s$.

Explain the meaning of the terms in these boundary conditions.

By writing $T = T_m + [T]\,\hat{T}$, choosing appropriate scales for the other variables, and linearising the surface energy balance (on the assumption that $[T]/T_m \ll 1$), derive the dimensionless model

$$\frac{1}{S}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} \quad b(t) < z < s(t), \qquad \hat{Q}(t) - T - \frac{\partial T}{\partial z} = m_s \quad \text{at} \quad z = s(t),$$

$$\frac{\partial T}{\partial z} + \hat{F}_o = m_b, \quad T = 0 \quad \text{at} \quad z = b(t), \qquad \dot{H}(t) = -m_b - m_s,$$

where $S = L/c[T]$ is the Stefan number, $\hat{F}_0 = F_0/4\sigma T_m^3[T]$ is the dimensionless ocean heat flux, and $\hat{Q}(t) = \left((1-a)Q - \sigma T_m^4\right)/4\sigma T_m^3[T]$ is the dimensionless radiative heat flux.

If $S \gg 1$, find the approximate temperature in the ice (assuming ice exists) and show that

$$\dot{H}(t) = \begin{cases} -\hat{F}_o - \hat{Q}/(1+H) & \hat{Q} \le 0 \quad \text{and} \quad H \ge 0, \\ -\hat{F}_o - \hat{Q} & \hat{Q} > 0 \quad \text{and} \quad H \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

[*The last case corresponds to there being no ice and the energy balance not being sufficient to grow any; in this case* $\hat{Q}$ *must fall below* $-\hat{F}_o$ *to transition to the first case and have the ice start to grow again.*]

If $\hat{Q} = \hat{Q}_0 + \cos\omega t$ is slowly varying (i.e. $\omega \ll 1$), find an approximate expression for the ice thickness $H(t)$, and hence find the fraction of time for which the sea is ice-covered as a function of $\hat{Q}_0$ (varying the value of $\hat{Q}_0$ corresponds to changing latitude).