## Extension sheet

[*These questions are intended to be attempted after the end of term, and are optional. Some require numerical solutions of the equations, for which you may write your own code or use the Matlab templates available online.*]

1. **Carbon cycles** Re-consider the model from Sheet 2 Question 3 for the evolution of albedo and partial pressure of atmospheric $CO_2$,

$$\dot{a} = f(a,p) = B(\Theta) - a,$$
$$\dot{p} = g(a,p) = \alpha(1 - wp^{\mu}e^{\Theta}),$$

where $\Theta(a,p) = \frac{q(1-a)-1}{\nu} + \lambda p$, and $B(\theta)$ is a monotonic function decreasing from $a_+$ to $a_-$, and where, $\mu$, $\alpha$, $\nu$, $\lambda$, $w$. and $q$ are all constant parameters.

Taking the specific form

$$B(\theta) = \tfrac{1}{2}(a_+ + a_-) + \tfrac{1}{2}(a_+ - a_-)\tanh\left(c_1 + c_2\theta\right),$$

with $a_- = 0.11$, $a_+ = 0.58$, $c_1 = 0.2$ and $c_2 = 0.6$, and taking other parameters as $\mu = 0.3$, $q = 1.37$, $\nu = 0.18$, $\lambda = 0.25$, solve the model numerically and confirm the results of the earlier question. That is, that a steady state on the intermediate branch of the $a$ nullcline is unstable if $\alpha$ is small enough.

Illustrate how the behaviour of the model depends on the parameters, but plotting example solutions for $a(t)$ and $p(t)$, and the trajectory $(p(t), a(t))$ on the phase plane.

2. **River mouth**. The water depth $h$ and velocity $u$ in a river are modelled using the dimensionless St Venant equations,

$$h_t + (hu)_x = 0, \qquad F^2(u_t + uu_x) = -h_x + 1 - \frac{u^2}{h},$$

where $F$ is the Froude number. The river has uniform depth $h = 1$ far upstream (where $x \to -\infty$). If the river flows into a large lake at $x = 0$ explain with a diagram why it may be appropriate to prescribe the condition

$$h \to x \quad \text{as} \quad x \to \infty.$$

Find an implicit expression for the steady-state water depth $h(x)$ in the case $F < 1$ (subcritical river flow), and draw a sketch of this solution.

If $F > 1$ (supercritical river flow), explain why the steady-state solution has $h = 1$ for $x < x_s$, and use jump conditions that conserve mass and momentum to determine the location of the shock,

$$x_s = -\tfrac{1}{2} + \tfrac{1}{2}(1+8F^2)^{1/2} - (F^2-1)\left[\frac{1}{3}\ln\left(\frac{-\frac{3}{2} + \frac{1}{2}(1+8F^2)^{1/2}}{(1+2F^2)^{1/2}}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{(1+8F^2)^{1/2}}\right)\right].$$

Draw a sketch of this solution.

[*Recall that the jump condition for a conservation equation* $P_t + Q_x = R$ *is* $\dot{x}_s = [Q]_-^+/[P]_-^+$.]

3. **Anti-dunes** A dimensionless model for stream flow over an erodible bed is given by

$$\frac{\partial}{\partial x}(hu) = 0, \qquad F^2 u\frac{\partial u}{\partial x} + \frac{\partial s}{\partial x} + \frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad \frac{\partial q}{\partial x} = q^*(u) - q,$$

where $s$ is the bed elevation, $h$ and $u$ are the water depth and velocity, $q^*(u)$ is a monotonically increasing bedload function, and $F$ is a constant.

Assuming that $s = 0$ and $h = u = 1$ at some point in the flow, find an algebraic relationship between $u$ and $s$, and show that the maximum possible value of $s$ is

$$s_* = 1 + \frac{1}{2}F^2 - \frac{3}{2}F^{2/3}.$$

Hence show that $q^*(u)$ can be treated as a multivalued function of $s$, with upper and lower branches denoted $f_+(s)$ and $f_-(s)$ respectively. Sketch a graph of this function, indicating the location of $s_*$.

Consider a travelling wave solution moving with constant velocity $-V$, where $V > 0$, and with bed elevation oscillating continuously between its minimum $s_{\rm min}$ and maximum $s_{\rm max}$. The bedslope is continuous except at $s_{\rm min}$. By writing $\xi = x + Vt$, and $s = s(\xi)$, show that

$$V\frac{\mathrm{d}s}{\mathrm{d}\xi} = Q - Vs - f_\pm(s),$$

where $Q = q + Vs$ is a constant.

By consideration of the sign of $\mathrm{d}s/\mathrm{d}\xi$ show that a solution which evolves smoothly from $s_{\rm min}$ to $s_{\rm max}$ and back again is only possible if $s_{\rm max} = s_*$ and $Q = Vs_* + f_\pm(s_*)$, and show that the wavelength of the solution is given by

$$\ell = \int_{s_{\rm min}}^{s_{\rm max}} \left\{ \frac{V}{Q - Vs - f_-(s)} - \frac{V}{Q - Vs - f_+(s)} \right\} \mathrm{d}s.$$

Draw a rough sketch of the travelling wave profiles for both the bed elevation and the corresponding water surface.

4. **Sea ice** Re-consider the model from Sheet 4 Question 4 for the seasonal evolution of sea ice,

$$\dot{H}(t) = \begin{cases} -\hat{F}_o - \hat{Q}/(1+H) & \hat{Q} \le 0 \quad \text{and} \quad H \ge 0, \\ -\hat{F}_o - \hat{Q} & \hat{Q} > 0 \quad \text{and} \quad H \ge 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $\hat{F}_o$ is the diemensionless ocean heat flux and $\hat{Q}(t)$ is the radiative flux.

Solve the ODE for ice thickness $H(t)$ numerically for the case $\hat{Q}(t) = \hat{Q}_0 + \cos\omega t$ but now with general $\omega$. Confirm your small–$\omega$ result from earlier.

[*You may need to be quite careful about the behaviour when $H$ decreases to zero.*]