Sheet 1

1. Lapse Rates $\quad \rho c_p \frac{dT}{dz} - \frac{dp}{dz} = 0 \quad \frac{dp}{dz} = -\rho g \quad p = \frac{\rho R T}{M_a}$ with $T = T_s$, $p = p_s$ at $z = 0$.

Combining $\Rightarrow \rho c_p \frac{dT}{dz} = -\rho g \Rightarrow \frac{dT}{dz} = -\frac{g}{c_p} \Rightarrow \boxed{T = T_s - \frac{g}{c_p} z}$ (note $\frac{g}{c_p} \approx 10$ K km^{-1})

Also $\frac{dp}{dz} = -\rho g = -\frac{M_a g}{RT} p \Rightarrow \frac{1}{p}\frac{dp}{dz} = \frac{M_a g}{R} \frac{1}{T_s - \frac{g}{c_p} z}$

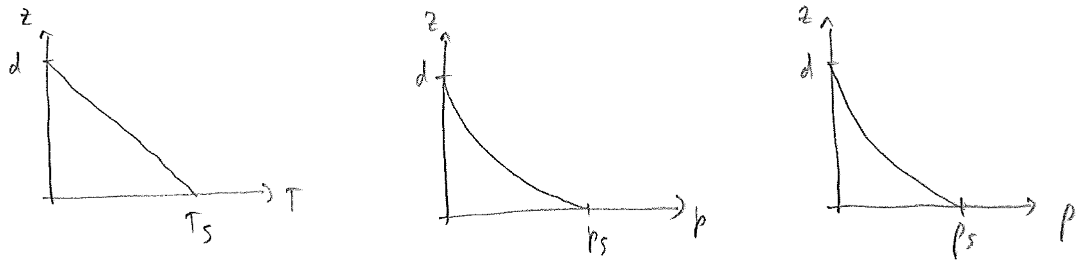
$$\Rightarrow \ln\left(\frac{p}{p_s}\right) = \frac{M_a c_p}{R} \ln\left(\frac{T_s - \frac{g}{c_p} z}{T_s}\right)$$

$$\Rightarrow \boxed{\frac{p}{p_s} = \left(\frac{T}{T_s}\right)^{\frac{M_a c_p}{R}}}$$

Then $\rho = \frac{M_a p}{RT} = \frac{M_a p_s}{R T_s}\left(\frac{p}{p_s}\right)\left(\frac{T_s}{T}\right) = \frac{M_a p_s}{R T_s}\left(\frac{T}{T_s}\right)^{\frac{M_a c_p}{R} - 1}$ (note $\frac{M_a c_p}{R} \approx 3.5$)

$$\Rightarrow \boxed{\rho = \underbrace{\frac{M_a p_s}{R T_s}}_{\rho_s} \left(\frac{T_s - \frac{g}{c_p} z}{T_s}\right)^{\frac{M_a c_p}{R} - 1}}$$

Since density, temp, and pressure all tend to zero at $z = d = \frac{c_p T_s}{g}$, this is a natural definition for the top of the atmosphere. If $T_s = 300$ K, this is around 30 km.



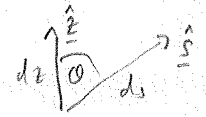
$\frac{\rho}{\rho_s} = \left(\frac{T}{T_s}\right)^{\frac{M_a c_p}{R} - 1}$ and $T = T_s - \frac{g}{c_p} z$. For Mt Everest, $z = 8{,}848$ m and $T_s \approx 300$ K, so

$\frac{T}{T_s} \approx \frac{2}{3}$ Hence $\frac{\rho}{\rho_s} \approx \left(\frac{2}{3}\right)^{2.5} \approx 0.36$, so around 30-40% thinner

Note: This question ignores the stratosphere, in which adsorption of shortwave radiation by ozone causes the temperature to rise again. The behaviour at large heights, including the prediction for the depth of the atmosphere, is therefore not correct. It is nevertheless interesting to see what the model predicts.

2. Two stream approximation

(i) $\boxed{\cos\theta \frac{\partial I}{\partial z} = -\kappa\rho(I-B)}$ $\quad B = \frac{\sigma T^4}{\pi}$



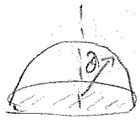
$dz = \cos\theta\, ds$

$\frac{\partial I}{\partial s} = -\kappa\rho(I-B)$ for $I(\underline{r}, \hat{s})$

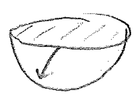
Write $\mu = \cos\theta$, so $\boxed{\mu \frac{\partial I}{\partial z} = -\kappa\rho(I-B)}$ (RTE)

Then define $I_+ = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} I \underbrace{\sin\theta\, d\theta}_{-d\mu}\, d\phi$

$= \int_0^1 I\, d\mu$

(i.e.  average intensity over upward directions)

and similarly $I_- = \frac{1}{2\pi}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} I \sin\theta\, d\theta\, d\phi = \int_{-1}^{0} I\, d\mu$



Then approximate $\int_0^1 \mu \frac{\partial I}{\partial z}\, d\mu \approx \frac{1}{2}\frac{dI_+}{dz}$ and $\int_{-1}^0 \mu \frac{\partial I}{\partial z}\, d\mu \approx -\frac{1}{2}\frac{dI_-}{dz}$,

and integrate RTE ($\int_0^1 d\mu$ and $\int_{-1}^0 d\mu$) to give

$$\boxed{\frac{1}{2}\frac{dI_+}{dz} = -\kappa\rho(I_+ - B) \qquad -\frac{1}{2}\frac{dI_-}{dz} = -\kappa\rho(I_- - B)}$$

Also, the total upward and downward radiative fluxes are given by

$F_+ = \int_0^{2\pi}\int_0^{\pi/2} I\cos\theta \sin\theta\, d\theta\, d\phi = 2\pi\int_0^1 \mu I\, d\mu \approx \pi I_+$

$F_- = \int_0^{2\pi}\int_{\pi/2}^{\pi} I\cos\theta \sin\theta\, d\theta\, d\phi = 2\pi\int_{-1}^0 \mu I\, d\mu \approx \pi I_-$

At $z=d$, (top of atmosphere), there is no incoming radiation, so $\boxed{I_- = 0 \text{ at } z=d}$

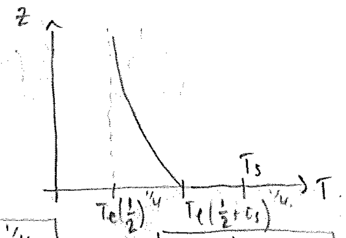
At $z=0$ (surface), Stefan-Boltzmann law gives $\boxed{I_+ = \frac{\sigma T_s^4}{\pi} \text{ at } z=0}$. Also $\boxed{I_+ = \frac{\sigma T_e^4}{\pi} \text{ at } z=d}$.

(ii) Local radiative equilibrium means $\boxed{B = \frac{1}{2}(I_+ + I_-)}$ so $\boxed{\frac{dI_+}{dz} = -\kappa\rho(I_+ - I_-) = \frac{dI_-}{dz}}$

Hence $\frac{d}{dz}(I_+ - I_-) = 0$ so $I_+ - I_-$ and hence $F_+ - F_- = F$ are constant. ($= \sigma T_e^4$ from $z=d$)

Then $\frac{dI_+}{dz} = \frac{dI_-}{dz} = -\frac{\kappa\rho F}{\pi} \Rightarrow \boxed{I_- = \frac{F}{\pi}\int_z^d \kappa\rho\, dz = \frac{F}{\pi}\tau}$ (using $I_- = 0$ at $z=d$)

& $\boxed{I_+ = \frac{F}{\pi}(1+\tau)}$



So $B = \frac{\sigma T^4}{\pi} = \frac{F}{\pi}\left(\frac{1}{2}+\tau\right) = \frac{\sigma T_e^4}{\pi}\left(\frac{1}{2}+\tau\right) \Rightarrow \boxed{T = T_e\left(\frac{1}{2}+\tau\right)^{1/4}}$

At $z=0$, $I_+ = \frac{\sigma T_s^4}{\pi} = \frac{F}{\pi}(1+\tau_s) = \frac{\sigma T_e^4}{\pi}(1+\tau_s) \Rightarrow \boxed{T_e = T_s(1+\tau_s)^{-1/4}}$ i.e. $\boxed{\gamma = \frac{1}{(1+\tau_s)^{1/4}}}$

(iii) If $T(z)$ is known, take $\frac{1}{2}\frac{dI_+}{dz} = -\kappa\rho\left(I_+ - \frac{\sigma T^4}{\pi}\right)$

Let $\tau = \int_z^d \kappa\rho\, dz$ so $\frac{d}{dz} = -\kappa\rho\frac{d}{d\tau}$. Then $\boxed{\frac{dI_+}{d\tau} - 2I_+ = -\frac{2\sigma T^4}{\pi}}$

Integrating factor $e^{-2\tau}$ $\Rightarrow$ $\frac{d}{d\tau}\left(I_+ e^{-2\tau}\right) = -\frac{\sigma T^4}{\pi}e^{-2\tau}$

$$\Rightarrow \quad I_+ e^{-2\tau} - \frac{\sigma T_s^4}{\pi}e^{-2\tau_s} = \int_\tau^{\tau_s} \frac{2\sigma T^4}{\pi} e^{-2\hat{\tau}}\, d\hat{\tau} \qquad \left(\text{using } I_+ = \frac{\sigma T_s^4}{\pi} \text{ at } \tau = \tau_s\right)$$

$$\Rightarrow \quad \boxed{I_+ = \frac{\sigma T_s^4}{\pi}\left[e^{-2(\tau_s - \tau)} + \int_\tau^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2(\hat{\tau}-\tau)}\, d\hat{\tau}\right]}$$

At $\tau = 0$ $(z = d)$, $I_+ = \frac{\sigma T_e^4}{\pi}$, so $T_e^4 = T_s^4\left[e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\hat{\tau}}\, d\hat{\tau}\right]$.

$$\text{ie.} \quad \boxed{\gamma = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\tau}\, d\tau}$$

(iv) From Q1, $\left(\frac{T}{T_s}\right) = \left(\frac{p}{p_s}\right)^{\frac{R}{M_a c_p}}$ and $p = \int_z^d \rho g\, dz$.

Note that $\tau = \int_z^d \kappa\rho\, dz$, so if κ is constant, then $\boxed{\tau = \frac{\kappa}{g}p}$ and hence $\frac{p}{p_s} = \frac{\tau}{\tau_s}$.

$$\text{So} \quad \boxed{\left(\frac{T}{T_s}\right) = \left(\frac{\tau}{\tau_s}\right)^{\frac{R}{M_a c_p}}}$$

$$\text{Hence} \quad \boxed{\gamma = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{\tau}{\tau_s}\right)^{\frac{4R}{M_a c_p}} e^{-2\tau}\, d\tau.}$$

For small τ_s, $\gamma = e^{-2\tau_s} + 2\tau_s\int_0^1 x^{\frac{4R}{M_a c_p}} e^{-2\tau_s x}\, dx$. $\quad [\tau = \tau_s x]$.

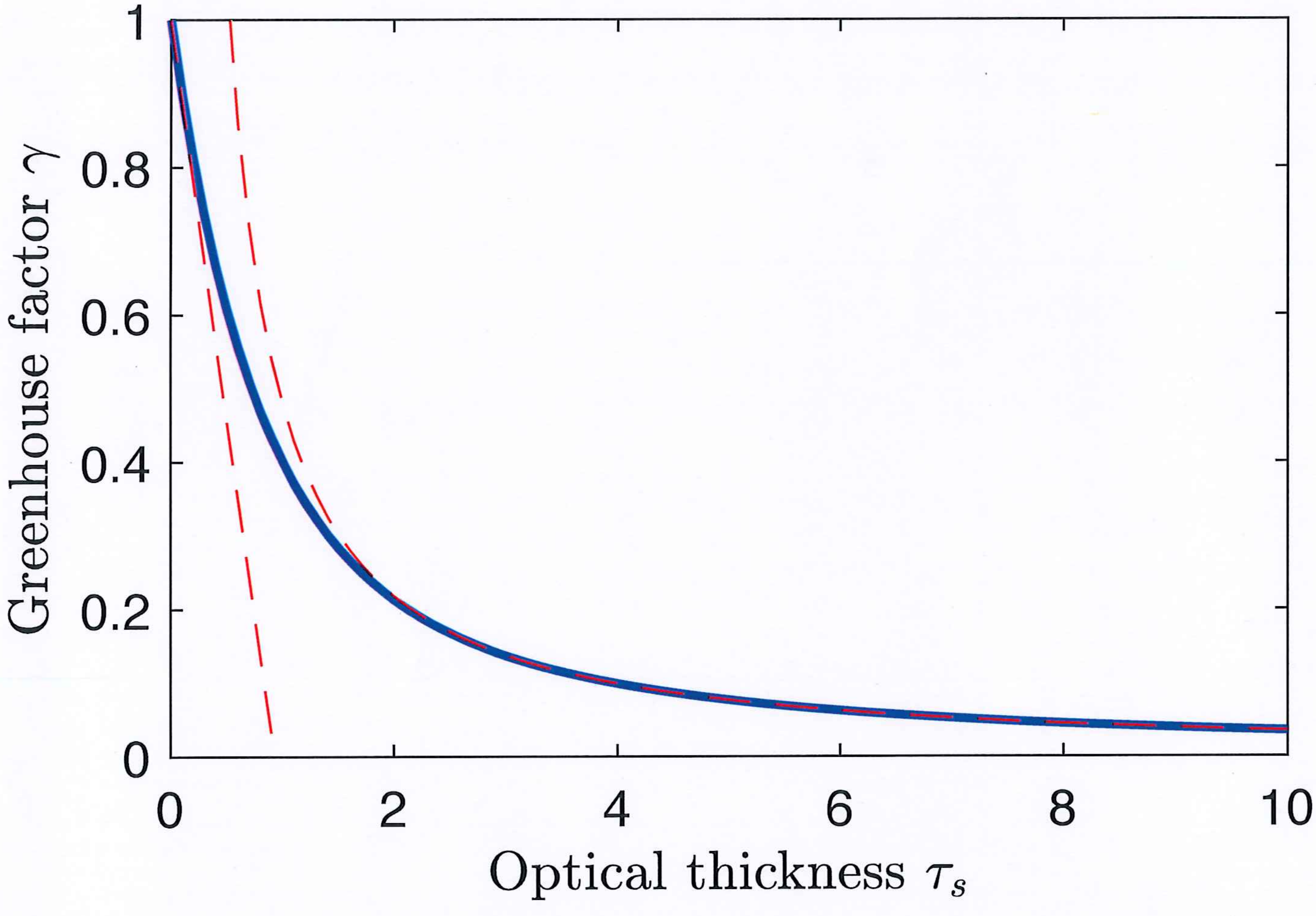
$$\sim 1 - 2\tau_s + \ldots + 2\tau_s\left(\int_0^1 x^{\frac{4R}{M_a c_p}}\, dx + O(\tau_s)\right) \qquad \int_0^1 x^\alpha\, dx = \frac{1}{\alpha+1}$$

$$= 1 - 2\tau_s + 2\tau_s\frac{M_a c_p}{4R + M_a c_p} + O(\tau_s^2) = \boxed{1 - \frac{8R}{4R + M_a c_p}\tau_s + O(\tau_s^2)}$$

For large τ_s, first term is exponentially small, and limit in integral can be replaced by ∞ with exponentially small errors, so $\gamma \sim \int_0^\infty \left(\frac{t}{2\tau_s}\right)^{\frac{4R}{M_a c_p}} e^{-t}\, dt$ $\quad [2\tau = t]$

$$= \boxed{(2\tau_s)^{-\frac{4R}{M_a c_p}} \underbrace{\int_0^\infty t^{\frac{4R}{M_a c_p}} e^{-t}\, dt}_{\Gamma\left(1 + \frac{4R}{M_a c_p}\right)}}$$

[See greenhouse_factor.m for numerical calculation of γ]



1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
Greenhouse factor γ
Optical thickness τ_s