Sheet 2

1. Running greenhouse effect.

Clausius Clapeyron equation $\quad \dfrac{\partial p_{sv}}{\partial T} = \dfrac{\rho_v L}{T} \qquad p_{sv} = p_{sv0}$ at $T = T_0$.

$$= \frac{p_{sv}}{T^2}\frac{M_v L}{R} \qquad \left[\text{using } p_{sv} = \frac{\rho_v R T}{M_v}\right]$$

[integrate] $\Rightarrow \quad \dfrac{dp_{sv}}{p_{sv}} = \dfrac{M_v L}{R}\dfrac{dT}{T^2}$

$$\ln\frac{p_{sv}}{p_{sv0}} = \frac{M_v L}{R}\left\{\frac{1}{T_0} - \frac{1}{T}\right\}$$

$$\Rightarrow \boxed{p_{sv} = p_{sv0}\, e^{a\left(1 - \frac{T_0}{T}\right)}} \qquad \boxed{a = \frac{M_v L}{R T_0}}$$

If $T - T_0 \ll T_0$ write $1 - \dfrac{T_0}{T} = 1 - \dfrac{1}{1 + \frac{T-T_0}{T_0}} = 1 - \left(1 - \dfrac{T-T_0}{T_0} + \dots\right) = \dfrac{T-T_0}{T_0} + O\left(\dfrac{T-T_0}{T_0}\right)^2$

Energy balance $\quad \dfrac{1}{4}Q = \sigma\gamma T^4 \quad \Rightarrow \boxed{T = \left(\dfrac{Q}{4\sigma\gamma}\right)^{\frac{1}{4}} = \left(\dfrac{Q}{4\sigma}\right)^{\frac{1}{4}}\left(1 + b\left(\dfrac{p_v}{p_{sv0}}\right)^c\right)}$

Write $\boxed{\theta = \dfrac{T}{T_0}}$, then energy balance gives $\quad \theta = \underbrace{\left(\dfrac{Q}{4\sigma T_0^4}\right)^{\frac{1}{4}}}_{\alpha}\left(1 + b\left(\dfrac{p_v}{p_{sv0}}\right)^c\right)$

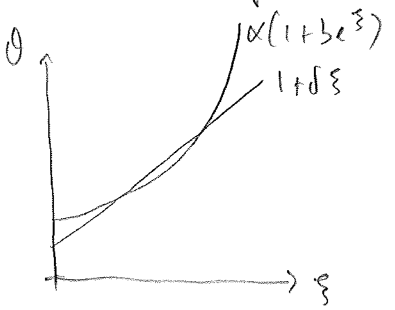
$$\Rightarrow \boxed{\theta = \alpha\left(1 + b e^{\xi_v}\right)} \quad \text{defining} \quad \boxed{\xi_v = c \ln\frac{p_v}{p_{sv0}}}$$

With the same notation, the saturation curve becomes. $\quad \dfrac{\xi_{sv}}{c} = a(\theta - 1)$

(but $\xi_{sv} = c \ln \dfrac{p_{sv}}{p_{sv0}}$)

$$\Rightarrow \boxed{\theta = 1 + \delta\xi_{sv}} \qquad \delta = \frac{1}{ac}.$$

The runaway greenhouse effect occurs if the temperature calculated from energy balance remains above the saturation curve for all p_v (ie ξ_v). So its occurrence depends on the non-intersection of $\theta = \alpha(1 + b e^{\xi})$ with $\theta = 1 + \delta\xi$.



Clearly from the graph, non intersection will happen if α is large enough.
The critical α is found from when the curves meet tangentially, ie. $\quad \boxed{\begin{aligned}\alpha(1 + b e^{\xi}) &= 1 + \delta\xi \\ \& \quad \alpha b e^{\xi} &= \delta.\end{aligned}}$

$$\Rightarrow \boxed{\alpha + \delta = 1 + \delta\xi = 1 + \delta\ln\left(\delta/\alpha b\right)}$$

If δ is small then, $\alpha \mathrel{\hat{=}} 1$, and putting this back into the right hand side gives improved estimate

$$\alpha \simeq 1 + \delta\ln\left(\delta/b\right) - \delta \qquad \left[\text{The corrections are } O(\delta^2 \ln\delta)\right]$$

For the Earth $Q = 1370\ Wm^{-2}$, $\sigma = 5.67 \times 10^{-8}\ Wm^{-2}K^{-4}$, $T_0 = 273\ K$.

So $\boxed{\alpha = \left(\frac{Q}{4\sigma T_0^4}\right)^{\frac{1}{4}} \approx 1.02119.}$

$R = 8.3\ J mol^{-1} K^{-1}$, $M_v = 18 \times 10^{-3}\ kg mol^{-1}$, $L = 2.5 \times 10^{6}\ J kg^{-1}$ $\Rightarrow$ $a \approx 19.9$.

$\Rightarrow$ $\delta \approx 0.2$ (using $c = 0.25$).

So $\boxed{\alpha_c \approx 1.05.}$

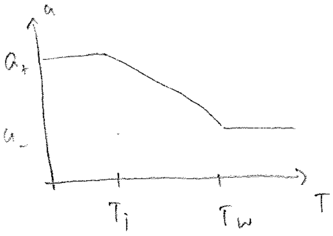
So $\alpha < \alpha_c$, suggesting runaway greenhouse effect does not occur.

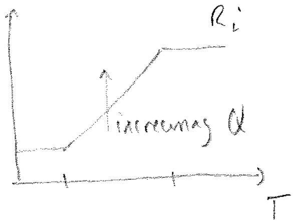
For Venus, Q is twice as large so α is increased by a factor of $2^{1/4}$ $\Rightarrow$ $\boxed{\alpha \approx 1.21}$

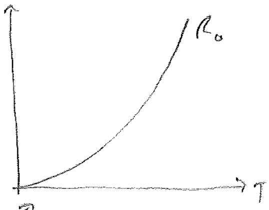
This is larger than α_c, so runaway greenhouse effect does occur. (i.e. the saturation vapour pressure is never reached)

x If solar radiation were 30% smaller when the atmospheres were forming, this would not make a difference, since decreasing α by $(0.7)^{1/4} \approx 0.91$ does not change the conclusion that $\alpha < \alpha_c$ for Earth and $\alpha > \alpha_c$ for Venus.

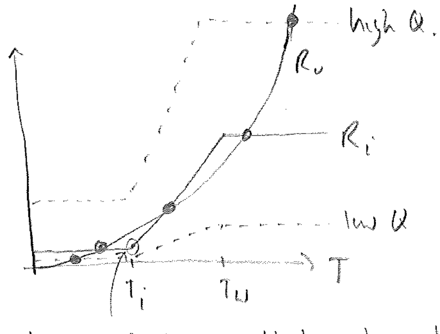
2. <u>Ice-albedo feedback.</u> $\quad c\dfrac{dT}{dt} = R_i - R_o. \qquad R_i = \frac{1}{4}Q(1-a) \qquad R_o = \sigma\gamma T^4$







Steady states are intersections of these

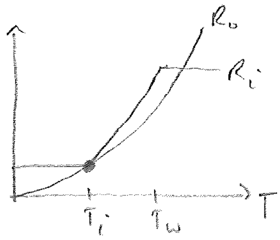


From the graph it is clear that there can be multiple intersections for intermediate values of Q, provided the slope of the central section of the R_i curve is sufficiently steep.

In particular, multiple intersections require $R_i(T_i) < R_o(T_i)$, and the slope $R_i'(T_i) = \frac{1}{4}Q\dfrac{a_+-a_-}{T_w-T_i}$ must be larger than $R_o'(T_i) = 4\sigma\gamma T_i^3$. The largest value that $R_i'(T_i)$ takes, while this point remains below the $R_o(T_i)$ curve is when $R_i(T_i) = R_o(T_i) \Rightarrow \frac{1}{4}Q(1-a_+) = \sigma\gamma T_i^4$

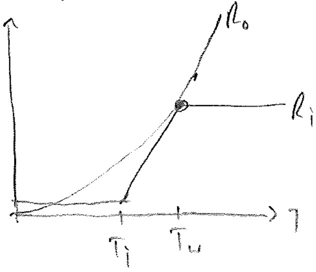
$\Rightarrow \Big| \; Q = \dfrac{4\sigma\gamma T_i^4}{1-a_+}$, so we require $\quad \dfrac{\sigma\gamma T_i^4}{1-a_+}\dfrac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_i^3$

$$\Leftrightarrow \quad \boxed{\frac{T_w-T_i}{T_i} < \frac{a_+-a_-}{4(1-a_+)}}$$

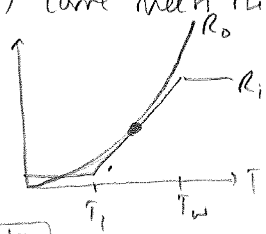


(equality would occur when the two slopes are equal)

If this condition on the slopes holds, then for smaller Q than this value $\left(\boxed{Q_+ = \dfrac{4\sigma\gamma T_i^4}{1-a_+}}\right)$ there will clearly be multiple intersections (as in diagram above). As Q is reduced, the multiple intersections cease to occur either when $R_i(T_w)$ drops below $R_o(T_w)$:



or when the $R_i(T)$ curve meets the $R_o(T)$ curve tangentially:



In the first case, the lower bound is $\boxed{Q_- = \dfrac{4\sigma\gamma T_w^4}{1-a_-}}$ and this applies if $R_i'(T_w) > R_o'(T_w)$ then

i.e. $\dfrac{\sigma\gamma T_w^4}{1-a_-}\dfrac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_w^3$

$$\Leftrightarrow \quad \boxed{\frac{T_w-T_i}{T_w} < \frac{a_+-a_-}{4(1-a_-)}}$$

In the second case, we must find the value of Q for which the curves meet tangentially.

This happens when $\frac{1}{4}Q(1-a) = \sigma\gamma T^4$, $\frac{1}{4}Q\frac{a_+-a_-}{T_w-T_i} = 4\sigma\gamma T^3$ $\Big\}$ solve for T and Q.

$1-a = 1-a_+ + \frac{a_+-a_-}{T_w-T_i}(T-T_i)$

Write $\lambda = \frac{a_+-a_-}{T_w-T_i}$, then $\frac{1}{4}Q(1-a_+ + \lambda(T-T_i)) = \sigma\gamma T^4$, $\frac{1}{4}Q\lambda = 4\sigma\gamma T^3$ $\Big\}$

$$1-a_+ + \lambda(T-T_i) = \frac{\lambda T}{4}.$$

$$\Rightarrow \frac{3}{4}T = T_i - \frac{(1-a_+)}{\lambda}$$

$$\Rightarrow T = \frac{4}{3}T_i - \frac{4}{3}\frac{(1-a_+)}{\lambda} = \frac{4}{3}\left[\frac{(1-a_-)T_i - (1-a_+)T_w}{a_+-a_-}\right]$$

Then $Q = \frac{16\sigma\gamma T^3}{\lambda} = \frac{16\sigma\gamma T^3(T_w-T_i)}{a_+-a_-} = \frac{64 \cdot 16\sigma\gamma}{27}\frac{(T_w-T_i)}{(a_+-a_-)^4}\left[(1-a_-)T_i - (1-a_+)T_w\right]^3.$

$$\Rightarrow \boxed{Q_- = \frac{\cancel{512}\,1024}{27}\sigma\gamma\frac{(T_w-T_i)\left[(1-a_-)T_i - (1-a_+)T_w\right]^3}{(a_+-a_-)^4}}$$

(This can be seen to give the same value as in the first case if $\frac{T_w-T_i}{T_w} = \frac{a_+-a_-}{4(1-a_+)}$)

3. Carbon cycles.

Start with dimensional model

$$c\dot{T} = \tfrac{1}{4}Q(1-a) - \sigma\gamma(p)T^4 \qquad a_0(T) = a_+ - \tfrac{1}{2}(a_+ - a_-)\left(1 + \tanh\left(\frac{T - T_m}{\Delta T}\right)\right)$$

$$t_i\dot{a} = a_0(T) - a. \qquad \gamma(p) = \gamma_0 - \gamma_1 p$$

$$\frac{M_{CO_2}A_E}{M_a g}\dot{p} = v - W(p,T). \qquad W = W_0\left(\frac{p}{p_0}\right)^\mu \exp\left(\frac{T - T_0}{\Delta T_c}\right)$$

Non-dimensionalise: $T = T_0 + \Delta T_c\theta$ $\quad p = p_0\hat{p}$ $\quad t = t_i\hat{t}$ $\quad$ (then drop hats.)

$$= T_0(1 + \tfrac{1}{4}\nu\theta) \qquad \nu = \frac{4\Delta T}{T_0}$$

$\Rightarrow$

$$\underbrace{\frac{c\Delta T_c}{t_i\sigma\gamma_0T_0^4}}_{\varepsilon}\dot{T} = \underbrace{\frac{Q}{4\sigma\gamma_0T_0^4}}_{q}(1-a) - (1 - \lambda\nu p)(1 + \tfrac{1}{4}\nu\theta)^4 \qquad \lambda = \frac{\gamma_1 p_0}{\gamma_0\nu}$$

$$w = \frac{W_0}{v}$$

$$\dot{a} = B(\theta) - a$$

$$\dot{p} = \underbrace{\frac{M_a g v t_i}{M_{CO_2}A_E p_0}}_{\alpha}(1 - wp^\mu e^\theta) \qquad B(\theta) = a_+ - \tfrac{1}{2}(a_+ - a_-)\left(1 + \tanh\left[\frac{T_0 - T_m}{\Delta T} + \frac{\Delta T_c\theta}{\Delta T}\right]\right)$$

$$\left[\text{i.e. } B(\theta) = a_0(T_0(1 + \tfrac{1}{4}\nu\theta))\right].$$

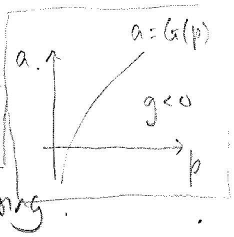
$\varepsilon \ll 1$ so take limit $\varepsilon \to 0$. Also ν is quite small, so expand nonlinear terms in small ν

$\longrightarrow$

$$\Rightarrow \quad 0 = q(1-a) - 1 + \lambda\nu p - \nu\theta \quad \Rightarrow \quad \boxed{\theta = \lambda p + \tfrac{1}{\nu}[q(1-a) - 1]}$$

together with the remaining equations

$$\boxed{\begin{aligned}\dot{a} &= B(\theta) - a \\ \dot{p} &= \alpha(1 - wp^\mu e^\theta).\end{aligned}} \quad \begin{aligned}&= f(a,p) \\ &= g(a,p)\end{aligned}$$



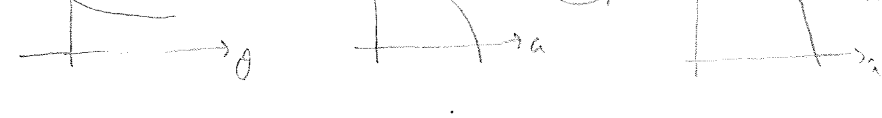
p nullcline: $\theta = -\ln w - \mu\ln p$

$$\theta = \lambda p + \tfrac{1}{\nu}(q(1-a) - 1) \quad \Rightarrow \quad \boxed{a = 1 - \frac{1}{q} + \nu(\lambda p + \mu\ln p + \ln w)}$$

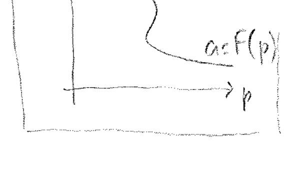
$= G(p)$ clearly monotonic increasing.

a nullcline: $a = B(\theta) = B\left(\lambda p + \frac{q(1-a) - 1}{\nu}\right)$ implicitly defines $a = F(p)$

$$\Rightarrow \boxed{\lambda p = \frac{1}{\nu} - \frac{q}{\nu} + \frac{q}{\nu}a + B^{-1}(a)} = F^{-1}(a)$$



positive slope if the linear term is large enough ie. if q large enough



We can see from the graphs that $p = F^{-1}(a)$ has a section of positive slope (and therefore $a = F(p)$ is multivalued) if q is large enough. In particular if $\frac{q}{\nu} > -\frac{d}{da} B^{-1}(a)$.

$$= -1/\frac{d}{d\theta}B(\theta)$$

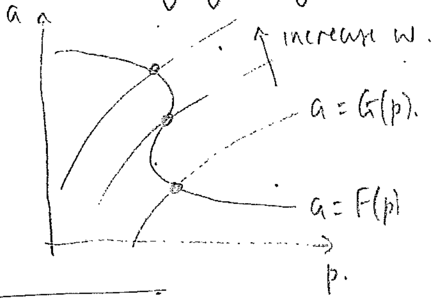
$$\iff \; -B'(\theta) > \frac{\nu}{q}.$$

If this never holds, $F^{-1}(a)$ is monotonic decreasing, so $a = F(p)$ is also.

[Alternatively, take $a = B(\theta)$ and differentiate implicitly $\frac{da}{dp} = B'(\theta) \cdot \left[-\frac{q}{\nu}\frac{da}{dp} + \lambda\right]$

$\Rightarrow \frac{da}{dp} = \frac{\lambda B'(\theta)}{1 + \frac{q}{\nu}B'(\theta)}$, which is always negative if $1 + \frac{q}{\nu}B'(\theta) > 0$ i.e. $-B'(\theta) < \frac{\nu}{q}$, but changes sign if $-B'(\theta) > \frac{\nu}{q}$ for any θ]

Note changing w just shifts the p nullcline up and down.

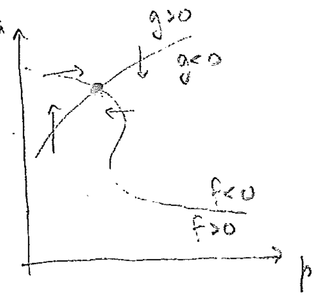


As w is increased the intersection pt. moves from the lower to the middle to the upper branch of the a nullcline.

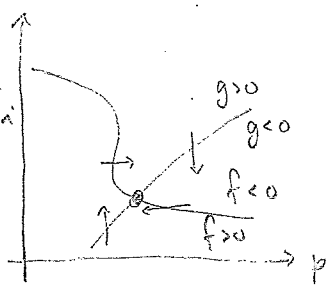
Note $f < 0$ to right of $a = F(p)$ (a nullcline). (so $\dot{a} < 0$ there)

$g < 0$ below $a = G(p)$ (p nullcline) (so $\dot{p} < 0$ there).

Upper branch intersection



Lower branch intersection

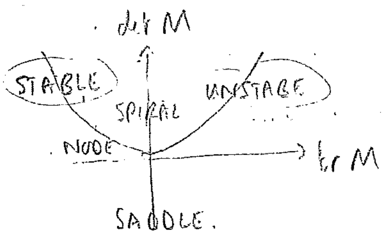


Linear stability depends on eigenvalues of

$$M = \begin{pmatrix} f_a & f_p \\ g_a & g_p \end{pmatrix} = \begin{pmatrix} <0 & <0 \\ >0 & <0 \end{pmatrix}$$

from looking at graphs.

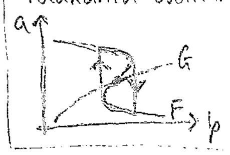
This has $\operatorname{tr} M < 0$ and $\det M > 0$ so these states are stable.



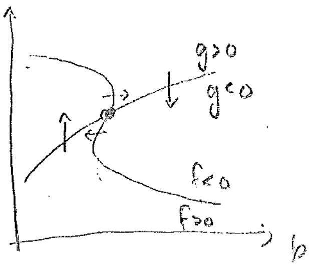
$\lambda^2 - \operatorname{tr} M \lambda + \det M = 0$.

complex $\lambda \iff (\operatorname{tr} M)^2 - 4 \det M < 0$

If $\alpha \ll 1$, have relaxation oscillations



Intermediate branch intersection.



different from before

$$M = \begin{pmatrix} >0 & <0 \\ >0 & <0 \end{pmatrix}$$

Now $\det M = f_a g_p - f_p g_a$

$$= \underbrace{f_a g_a}_{>0}\left(\frac{g_p}{g_a} - \frac{f_p}{f_a}\right) = f_a g_a\big(\underbrace{-G'(p)}_{>0} + \underbrace{F'}_{>0}$$

$\operatorname{tr} M = f_a + g_p > 0$ if α sufficiently small (since g is proportional to α)

4. Ocean Carbon.

(1) $c\frac{dT}{dt} = \frac{1}{4} Q(1-a) - \sigma\gamma(p) T^4$ Radiative balance (shortwave from sun, longwave + greenhouse factor)

(2) $t_i \frac{da}{dt} = a_0(T) - a$ Growth and shrinkage of ice sheets controls albedo (phenomenological)

(3) $\frac{A_E M_{CO_2}}{g M_a}\frac{dp}{dt} = v - h(p - p_s)$ Conservation of CO_2 in atmosphere (v = emissions, $h(p-p_s)$ = absorption into ocean)

(4) $\rho_0 V_0 \frac{dC}{dt} = \frac{h(p-p_s)}{M_{CO_2}} - bC$ Conservation of inorganic carbon in ocean ($\frac{h(p-p_s)}{M_{CO_2}}$ = exchange with atmosphere, bC = uptake by plankton → sedimentation)

(5) $p_s = \frac{C}{k}$ Henry's law - equilibrium vapour pressure.

Estimate timescales:

(1) $t_T = \frac{cT_0}{Q(1-a)}$ T_0 from $\frac{1}{4} Q(1-a) = \sigma\gamma T_0^4 \to T_0 \sim 290\,K$

$\hookrightarrow t_T \sim \frac{10^7 \; 290}{1370 \cdot 0.7} \; \frac{J m^{-2} K^{-1} K}{W m^{-2}} \sim \frac{10^7 \; 3 \; 10^2}{10^3} s \sim \boxed{0.1\,y}$

(2) $t_a = t_i = 10^4\,y$ (given)

[$Pa = kg\,m^{-1}s^{-2}$]

(3) $t_p = \frac{A_E M_{CO_2}}{g M_a h} \sim \frac{5.1\times10^{14} \; 44\cdot10^{-3}}{9.8 \;\; 29\cdot10^{-3} \; 0.7\cdot10^{12}} \; \frac{m^2 \; \cancel{kg\,mol^{-1}}}{m s^{-2} \; \cancel{kg\,mol^{-1}} \; kg\,y^{-1} Pa^{-1}} \sim \boxed{10^2\,y}$

(4) $t_C = \frac{\rho_0 V_0}{b} \sim \frac{10^3 \; 1.35\;10^{18}}{0.83\cdot10^{16}} \; \frac{kg\,m^{-3}\,m^3}{kg\,y^{-1}} \sim \boxed{10^5\,y}$

So p and T evolve considerably faster than a and C → treat them as quasi-steady. So

$\boxed{T = \left(\frac{Q(1-a)}{4\sigma\gamma(p)}\right)^{1/4}}$ and $\boxed{p = p_s + \frac{v}{h} = \frac{C}{k} + \frac{v}{h}}$

and (2), (4) become (2) $t_i \dot{a} = a_0(T) - a$
(4) $t_C \dot{C} = C_v - C$ where $C_v = \frac{v}{M_{CO_2} b}$

If present day $C \approx 2\times10^{-3}$ mol kg^{-1} assumed to be in equilibrium with pre-industrial emissions (C evolves slowly so has changed little since then) then $C_v = 2\times10^{-3}$ mol kg^{-1}

$\Rightarrow v \approx 2\times10^{-3} \cdot 44\times10^{-3} \; 0.83 \; 10^{16}$ ~~mol kg^{-1} kg mol^{-1}~~ kg y^{-1} $\approx \boxed{7\times10^{11}\,kg\,y^{-1}}$

then $p = \frac{C}{k} + \frac{v}{h} \approx \frac{2\times10^{-3}}{7.1\cdot10^{-5}} + \cancel{\frac{7\times10^{11}}{0.73\cdot10^{12}}} \approx \boxed{28\,Pa}$ [~280 ppm, about right]

negligible in comparison with first term

If present day emissions $V \approx 30 \times 10^{12}$ kg y^{-1} maintained indefinitely then p adjusts on timescale of centuries to the new quasi-equilibrium. $p = \dfrac{C}{K} + \dfrac{V}{h} \approx 28 + 41$ Pa

$\approx \boxed{69 \text{ Pa}}$ (≈ 700 ppm)

(on this timescale C remains roughly constant).

Then on the longer timescale t_C (millennial timescale) C evolves towards C_V (according to equation (4)), which is now $\approx \dfrac{30 \; 10^{12}}{44 \; 10^{-3} \; 0.83 \; 10^{16}} \approx 10^{-1}$ mol kg^{-1}

On this longer timescale, p evolves quasistatically towards $p = \dfrac{C_V}{K} + \dfrac{V}{h}$

$\approx \boxed{1000 \text{ Pa.}}$ ($\approx 10{,}000$ ppm)