Sheet 3

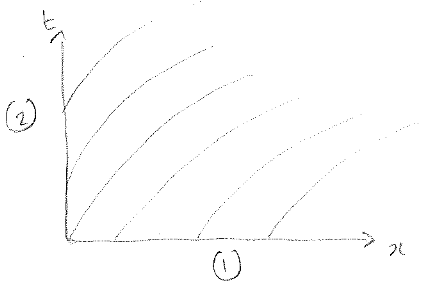
1. Overland flow

$$h_t + ch^m h_x = E$$

Characteristics: $\dot{x} = ch^m \quad \dot{h} = E \qquad \dot{} = \frac{d}{dt}$

with initial data (1) $x = x_0 \quad h = 0 \quad$ at $t = 0$.

(2) $x = 0 \quad h = 0 \quad$ at $t = t_0$



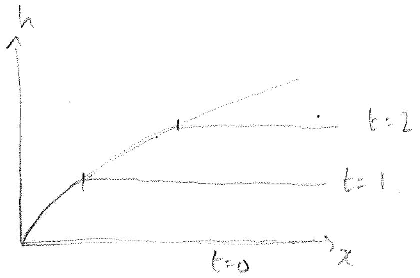
For characteristics from (1) $\quad h = Et \qquad x = \frac{cE^m}{m+1} t^{m+1} + x_0$

" $\quad$ (2) $\quad h = E(t - t_0) \qquad x = \frac{cE^m}{m+1}(t - t_0)^{m+1}$

$$= E^{1-\frac{m}{m+1}} \left(\frac{m+1}{c}\right)^{\frac{1}{m+1}} x^{\frac{1}{m+1}}$$

$\boxed{h = Et.}$ for $x > \frac{cE^m t^{m+1}}{m+1}$

$\boxed{h = \left(\frac{E(m+1)}{c}\right)^{\frac{1}{m+1}} x^{\frac{1}{m+1}}}$ for $x < \frac{cE^m t^{m+1}}{m+1}$.



Note h should depend only on time (not x) for $x > \frac{cE^m}{m+1} t^{m+1}$ and only on x (not t) for $x < \frac{cE^m}{m+1} t^{m+1}$.

For $E(t)$ follow same characteristic solution:

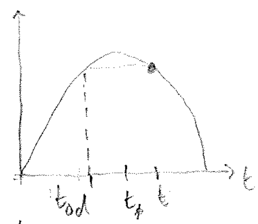
Now (1) $\boxed{h = \int_0^t E(\hat{t})\,d\hat{t}} \qquad x = x_0 + \int_0^t c\left(\int_0^{t'} E(\hat{t})\,d\hat{t}\right)^m dt'$

(2) $h = \int_{t_0}^t E(\hat{t})\,d\hat{t} \qquad x = \int_{t_0}^t c\,\Big(\underbrace{\int_{t_0}^{t'} E(\hat{t})\,d\hat{t}}_{H(t') - H(t_0)}\Big)^m dt'$

Define $\boxed{H(t) = \int_0^t E(\hat{t})\,d\hat{t}}$ (see diagram).

This implicitly defines t_0 in terms of x and t. Then this determines h.

$H(t) = \int_0^t E(\hat{t})\,d\hat{t}$



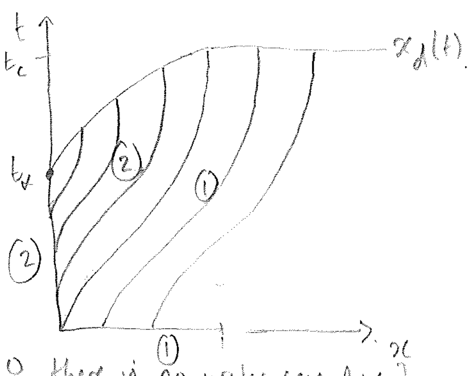
For $t > t_*$ the same solution holds where there is water $(x > x_d)$ but there is a drying front $x_d(t)$ at which $h = 0$ and behind which there is no water. $(h = 0)$.

At time t, the characteristic that intersects the drying front is identified from $\quad h = 0 = \int_{t_0}^t E(\hat{t})\,d\hat{t} \to t_0 \quad$ (call it $t_{0d}(t)$)

(see diagram above).

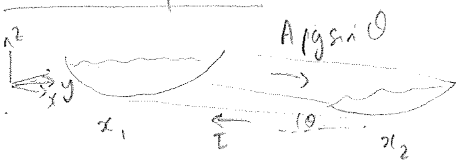
Then x_d determined from the position of that characteristic:

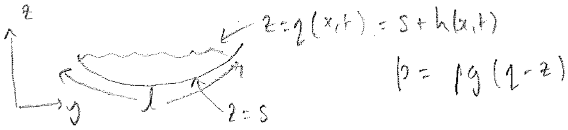
$$\boxed{x_d = \int_{t_{0d}(t)}^t c\,\left(H(t') - H(t_{0d}(t))\right)^m dt'}$$



[For $t > t_c$, where $H(t_c) = 0$, there is no water anywhere]

2. St Venant equations





Mass conservation $\frac{\partial}{\partial t}\int_{x_1}^{x_2} A\,dx = Q|_{x_1} - Q|_{x_2} = -\int_{x_1}^{x_2}\frac{\partial Q}{\partial x}\,dx$ where $\boxed{Q = Au}$ is volume flux.
(water is incompressible so volume is conserved).

Since x_1, x_2 are arbitrary, this implies $\boxed{\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0.}$ (1)
(& assuming continuity)

Momentum conservation $\frac{\partial}{\partial t}\int_{x_1}^{x_2}\rho A u\,dx = \underbrace{\rho A u^2|_{x_1} - \rho A u^2|_{x_2}}_{\text{momentum fluxes}} - \underbrace{\int_{x_1}^{x_2}\tau l}_{\text{friction}} + \underbrace{\rho g A \sin\theta}_{\text{weight}} + \underbrace{\bar{p}A|_{x_1} - \bar{p}A|_{x_2}}_{\text{pressure forces}}$.

$S = \sin\theta$

$$= \int_{x_1}^{x_2} -\frac{\partial}{\partial x}(Au^2) - \tau l + \rho g A S - \frac{\partial}{\partial x}(\bar{p}A)\,dx.$$

Since x_1, x_2 are arbitrary, $\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}(\rho A u^2) = \rho g A S - \tau l - \frac{\partial}{\partial x}(\bar{p}A)$

Since $p = \rho g(\eta - z)$, $\bar{p}A = \int_A \rho g(\eta - z)\,dy\,dz$ $\quad \frac{\partial}{\partial x}(\bar{p}A) = \int_A \rho g \eta_x\,dy\,dz = \rho g A \eta_x$, assuming η_x is independent of y. Also assume s flat, so $\eta_x = \bar{h}_x$, $\bar{h}$ = average depth.

Using (1), dividing by ρA $\Rightarrow \boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{\tau l}{\rho A} - g\bar{h}_x.}$ (2).

Steady uniform flow would have $gS = \frac{\tau l}{\rho A}$, $\Rightarrow$ $\tau = \rho g S R$ where $R = \frac{A}{l}$

If this has $u = \frac{R^{2/3}S^{1/2}}{n}$ then (eliminating S) $\boxed{\tau = \frac{\rho g n^2}{R^{1/3}}u^2}$

For a triangular cross-section, $h = \frac{l}{2}\sin\beta$, $A = \left(\frac{l}{2}\right)^2\sin\beta\cos\beta = \frac{1}{8}l^2\sin(2\beta)$. $\Rightarrow$ $l = A^{1/2}\left(\frac{\sin 2\beta}{8}\right)^{-1/2}$

so $R = \frac{A}{l} = \frac{1}{8}l\sin 2\beta = \left(\frac{\sin 2\beta}{8}\right)^{1/2}A^{1/2}$ and $\bar{h} = \frac{\sin\beta}{4}\left(\frac{\sin 2\beta}{8}\right)^{-1/2}A^{1/2}$.

Hence $\boxed{\tau = \rho g n^2\left(\frac{\sin 2\beta}{8}\right)^{-1/6}\frac{u^2}{A^{1/6}}}$ and $\boxed{\bar{h} = \left(\frac{\tan\beta}{4}\right)^{1/2}A^{1/2}}$ $\quad \frac{\tau l}{\rho A} = \frac{\tau}{\rho R} = \frac{g n^2}{R^{4/3}}u^2 = g n^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}\frac{u^2}{A^{2/3}}$

Substituting into (2), we have $\quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - gn^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}\frac{u^2}{A^{2/3}} - g\left(\frac{\tan\beta}{4}\right)^{1/2}\frac{\partial}{\partial x}(A^{1/2})$

Non-dimensionalise: $x = L\hat{x}$, $A = [A]\hat{A}$, $u = [u]\hat{u}$, $t = [t]\hat{t}$

$$gS = gn^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}\frac{[u]^2}{[A]^{2/3}} \quad \& \quad [A][u] = Q \quad \to [u],\,[A]$$

$$[t] = \frac{L}{[u]}. \quad \to [t]$$

Solving for $[u]$ $\Rightarrow$

$$SQ^{2/3} = n^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}[u]^{8/3} \;\Rightarrow\; [u] = S^{3/8}Q^{1/4}\left(\frac{\sin 2\beta}{8}\right)^{1/4}n^{-3/4}.$$

$$[A] = Q^{3/4}S^{-3/8}\left(\frac{\sin 2\beta}{8}\right)^{-1/4}n^{3/4}.$$

Then non-dimensional equations are

$$\boxed{\begin{aligned}&\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0\\ &\varepsilon F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A^{2/3}} - \varepsilon\frac{\partial}{\partial x}(A^{1/2})\end{aligned}}$$

where $\boxed{\varepsilon = \left(\frac{\tan\beta}{4}\right)^{1/2}\frac{[A]^{1/2}}{LS}} = \frac{[h]}{LS}$ and $\boxed{F^2 = \frac{[u]^2}{g[h]}}$ $\quad \left([h] = \left(\frac{\tan\beta}{4}\right)^{1/2}[A]^{1/2}\right)$

- If $\varepsilon \ll 1$ and $F \ll 1$, to leading order $A^{2/3} = u^2 \Rightarrow u \approx A^{1/3}$.

Then to next order in ε, $u^2 = A^{2/3} - \varepsilon A^{2/3}\frac{\partial}{\partial x}(A^{1/2}) = A^{2/3} - \frac{1}{2}\varepsilon A^{1/6}\frac{\partial A}{\partial x}$

$$\Rightarrow u = A^{1/3}\left(1 - \frac{\varepsilon}{2}A^{-1/2}\frac{\partial A}{\partial x}\right)^{1/2}$$

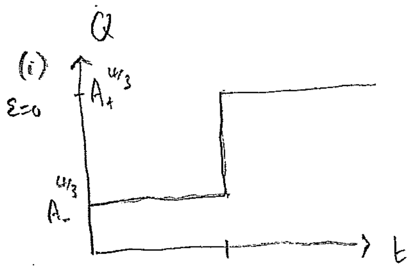
$$\approx A^{1/3}\left(1 - \frac{\varepsilon}{4}A^{-1/2}\frac{\partial A}{\partial x} + \ldots\right) = A^{1/3} - \frac{\varepsilon}{4}A^{-1/6}\frac{\partial A}{\partial x}.$$

So mass equation becomes

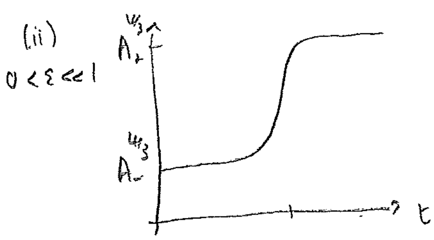
$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left[A^{4/3} - \frac{\varepsilon}{4}A^{5/6}\frac{\partial A}{\partial x}\right] = 0$$

or

$$\boxed{\frac{\partial A}{\partial t} + \frac{4}{3}A^{1/3}\frac{\partial A}{\partial x} = \frac{\varepsilon}{4}\frac{\partial}{\partial x}\left(A^{5/6}\frac{\partial A}{\partial x}\right)}$$



Shock.



Smoothed-out shock.

3. Surface waves

As in previous question, and choosing $L = [h]/S$ so as to make $\varepsilon = 1$, the equations for a triangular-shaped cross-section with Manning's law take the form

$$\boxed{\begin{aligned} &\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0 \\ &F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A^{2/3}} - \frac{1}{2A^{1/2}}\frac{\partial A}{\partial x} \end{aligned}} \qquad F^2 = \frac{[u]^2}{g[h]}.$$

Uniform steady state has $u^2 = A^{2/3}$ and if discharge is 1, $Au = 1$

$$\Rightarrow u = A = 1.$$

Perturbations $u = 1 + U$, $A = 1 + a$ with $U, a \ll 1$. Linearized equations

$$\boxed{\begin{aligned} &\frac{\partial a}{\partial t} + \frac{\partial a}{\partial x} + \frac{\partial U}{\partial x} = 0. \\ &F^2\left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\right) = -2U + \frac{2}{3}a - \frac{1}{2}\frac{\partial a}{\partial x}. \end{aligned}} \quad \rightarrow \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)a = -\frac{\partial U}{\partial x}$$

$$\Rightarrow \quad F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 U = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)U - \frac{2}{3}\frac{\partial U}{\partial x} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}$$

Solutions of form $U = e^{\frac{\sigma t + ikx}{F^2}}$ here $\quad \dfrac{F^2(\sigma + ik)^2}{F^4} = -\dfrac{2(\sigma + ik)}{F^2} - \dfrac{2}{3}\dfrac{ik}{F^2} - \dfrac{1}{2}\dfrac{k^2}{F^4}$

$$\Rightarrow \quad (\sigma + ik)^2 + 2(\sigma + ik) + \frac{2}{3}ik + \frac{1}{2}\frac{k^2}{F^2} = 0$$

$$\sigma + ik = -1 \pm \underbrace{\left(1 - \frac{2}{3}ik - \frac{k^2}{2F^2}\right)^{1/2}}_{p + iq}$$

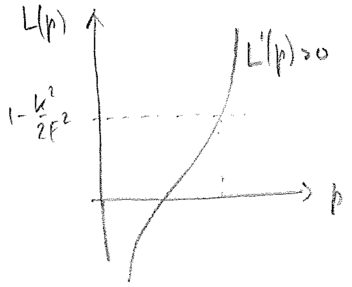
$(p + iq)^2 = 1 - \frac{2}{3}ik - \frac{k^2}{2F^2}$ $\quad$ ($p > 0$ w.l.o.g.)

$\Rightarrow pq = -\frac{1}{3}k \quad \rightarrow \quad \boxed{q = -k/3p}$

& $p^2 - q^2 = 1 - \frac{k^2}{2F^2} \quad \rightarrow \quad \boxed{p^2 - \frac{k^2}{9p^2} = 1 - \frac{k^2}{2F^2}}$

$$\underbrace{p^2 - \frac{k^2}{9p^2}}_{= L(p)}$$

$$\Rightarrow \quad \boxed{\sigma = -1 \pm p - ik\left(1 \pm \tfrac{1}{3p}\right)}$$



Velocity of perturbations is $-\frac{\sigma_I}{k} = 1 \pm \frac{1}{3p}$

propagate up and downstream if $p < \frac{1}{3} \iff L(p) < L(\frac{1}{3})$

$$\iff 1 - \frac{k^2}{2F^2} < \frac{1}{9} - k^2$$

$$\iff k^2\left(\frac{1}{2F^2} - 1\right) > \frac{8}{9}$$

This occurs for at least some k, if $F^2 < \frac{1}{2}$. ie. $\boxed{F < \frac{1}{\sqrt{2}} = F_1}$

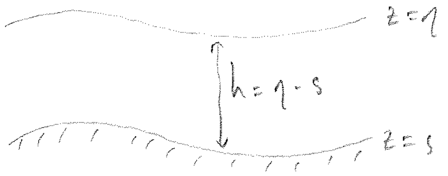
Otherwise $p > \frac{1}{3}$ and all waves propagate downstream.

Instability occurs if $\sigma_R > 0 \iff p > 1 \iff L(p) > L(1)$

$$\iff 1 - \frac{k^2}{2F^2} > 1 - \frac{k^2}{9}$$

$$\iff F^2 > \frac{9}{2} \qquad \text{ie. } \boxed{F > \frac{3}{\sqrt{2}} = F_2}$$

4. Dunes and antidunes



$\varepsilon h_t + (hu)_x = 0$ Mass conservation

$F^2(\varepsilon u_t + u u_x) = -\eta_x + \delta\left(1 - \frac{u^2}{h}\right)$ Momentum conservation. (terms on right: pressure gradient, downslope gravity, friction)

$h(\varepsilon c_t + u c_x) = E(u) - c = -s_t$. Suspended sediment conservation + Exner equation with erosion rate $E(u)$, deposition rate c (no bedload transport)

F is the Froude number, describes how fast the flow is.
ε represents ratio of advection timescale (for water) to bed evolution timescale.
δ represents importance of gravity and friction to force balance on the scale of a dune.

Steady state $u=h=c=1$, $s=0$ perturbed by writing $s=S$, $u=1+U$, $h=1+H$, $c=1+C$,
then with $\varepsilon = 0$,

$H + U = 0 \qquad \rightarrow H = -U$

$F^2 U_x = -H_x - S_x + \delta(H - 2U) \qquad (F^2-1)U_x + 3\delta U = -S_x$

$C_x = \cancel{E(1) - 1} + E'(1)U - C = -S_t \qquad C_x = E'(1)U - C = -S_t$.

Suppose $U = e^{\sigma t + ikx}$ $\quad C = \hat{C} e^{\sigma t + ikx}$ $\quad S = \hat{S} e^{\sigma t + ikx}$. Then,

$ik\hat{C} = E'(1) - \hat{C} \qquad \rightarrow \hat{C} = \frac{E'(1)}{1+ik}$

$ik(F^2-1) + 3\delta = -ik\hat{S} \qquad \rightarrow -\hat{S} = \frac{ik(F^2-1)+3\delta}{ik}$

$ik\hat{C} = -\sigma\hat{S}$

$$\Rightarrow \sigma = \frac{ik\hat{C}}{-\hat{S}} = \frac{E'(1)\,ik}{(1+ik)\left((F^2-1) - 3i\delta/k\right)}$$

$$= \frac{E'(1)\,ik(1-ik)\left((F^2-1)+3i\delta/k\right)}{(1+k^2)\left[(F^2-1)^2 + 9\delta^2/k^2\right]}$$

$$= \frac{E'(1)\left\{\left[(F^2-1)k^2 - 3\delta\right] + ik\left[(F^2-1)+3\delta\right]\right\}}{(1+k^2)\left[(F^2-1)^2 + 9\delta^2/k^2\right]}.$$

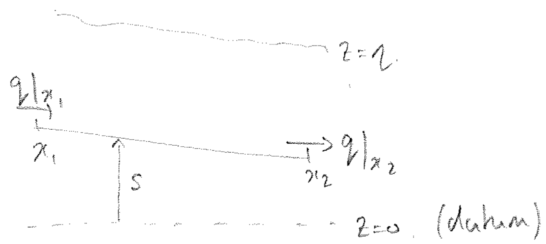
If $F>1$, $\mathrm{Re}\,\sigma > 0$ for $k^2 > \frac{3\delta}{F^2-1}$ so unstable.
$-\mathrm{Im}\frac{\sigma}{k} < 0$ in this case so waves move upstream. $\Big\} \rightarrow$ corresponds to growth of antidunes

If $F<1$, $\mathrm{Re}\,\sigma < 0$ so system is stable.
$-\mathrm{Im}\frac{\sigma}{k}$ has sign depending on δ and F.

Note $\eta - 1 = H + S = (\hat{S}-1)e^{\sigma t + ikx} = \frac{\hat{S}-1}{\hat{S}} S$,
$\quad \frac{\hat{S}-1}{\hat{S}} = 1 + \frac{1}{(F^2-1) - 3i\delta/k} = 1 + \frac{(F^2-1) + \frac{3i\delta}{k}}{(F^2-1)^2 + \frac{9\delta^2}{k^2}}$

The argument of this factor gives the phase difference between surface and bed.
If $\boxed{\frac{\delta}{k} \ll 1}$, then it is $\boxed{\frac{F^2}{F^2-1}}$, so suggests in phase if $F>1$, out of phase if $F<1$. If $\boxed{\frac{\delta}{k} \gg 1}$ it is $\boxed{1 + \frac{ik}{3\delta} + \ldots \approx 1}$
so waves are in phase.

5. <u>Eddy-viscosity model</u>



Consider a section of the bed between x_1 and x_2.

$$\frac{\partial}{\partial t}\left(\int_{x_1}^{x_2}(1-n)S\,dx\right) = q|_{x_1} - q|_{x_2} = -\int_{x_1}^{x_2}\frac{\partial q}{\partial x}\,dx$$

rate of change of volume of sediments in the section.

Since x_1, x_2 are arbitrary, $\boxed{(1-n)\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0}$ assume porosity n is constant.

If $q = q(\tau)$. $\quad (1-n)\frac{\partial s}{\partial t} + q'(\tau)\frac{\partial \tau}{\partial x} = 0$

and if we scale t to $[t]$ such that $\quad [t] = \frac{(1-n)[s][x]}{[q]}$ the equation becomes

scales for $[s], [x], [q]$ all arbitrary at this stage.

$$\boxed{\frac{\partial s}{\partial t} + q'(\tau)\frac{\partial \tau}{\partial x} = 0.}$$

Dimensionless shear stress model: $\tau = 1 - s + \int_0^\infty k(\xi)\frac{\partial s}{\partial x}(x-\xi, t)\,d\xi$ for $k(\xi) = \frac{\mu}{\xi^{1/3}}$.

Perturb the steady state $s = 0 + S$, $\tau = 1 + T$, then

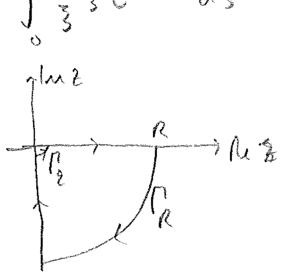
$$\boxed{\frac{\partial S}{\partial t} + q'(1)\frac{\partial T}{\partial x} = 0.} \quad \boxed{T = -S + \int_0^\infty \frac{\mu}{\xi^{1/3}}\frac{\partial S}{\partial x}(x-\xi, t)\,d\xi}$$

Look for solutions of form $S = e^{\sigma t + ikx}$ $\quad T = \hat{T}e^{\sigma t + ikx}$, then

$$\boxed{\sigma = -ikq'(1)\hat{T}}, \quad \boxed{\hat{T} = -1 + ik\int_0^\infty \frac{\mu}{\xi^{1/3}}e^{-ik\xi}\,d\xi.} = -1 + \mu\underbrace{e^{-i\pi/3}}_{\frac{1}{2}-\frac{\sqrt{3}}{2}i}ik^{1/3}\Gamma\left(\tfrac{2}{3}\right) \quad \text{(using hint)}.$$

$$\int_0^\infty \frac{1}{\xi^{1/3}}e^{-ik\xi}\,d\xi = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\Bigg[-\underbrace{\int_\Gamma z^{-1/3}e^{-ikz}\,dz}_{0 \text{ by Cauchy's theorem}} + \underbrace{\int_{\Gamma_R} z^{-1/3}e^{-ikz}\,dz}_{\to 0 \text{ by Jordan's lemma}} + \int_0^{-iR} z^{-1/3}e^{-ikz}\,dz - \underbrace{\int_{\Gamma_\varepsilon} z^{-1/3}e^{-ikz}\,dz}_{\to 0 \text{ by estimation lemma.}}$$

$z = \frac{-it}{k}$



$$= -0 + 0 + \left(-\frac{i}{k}\right)^{2/3}\underbrace{\int_0^\infty t^{-1/3}e^{-t}\,dt}_{\Gamma(\frac{2}{3})} - 0 \quad \text{(assuming } k>0\text{)}.$$

$$= \frac{e^{-i\pi/3}}{k^{2/3}}\Gamma\left(\tfrac{2}{3}\right)$$

So $\boxed{\sigma = \frac{1}{2}\mu q'(1)\Gamma\left(\tfrac{2}{3}\right)k^{4/3} - ikq'(1)\left[\frac{\sqrt{3}}{2}\mu q'(1)\Gamma\left(\tfrac{2}{3}\right)k^{4/3} - 1\right]}$

Perturbations are unstable ($\sigma_R > 0$) assuming $q'(1) > 0$. Perturbation travel downstream $\left(\frac{-\sigma_I}{k} > 0\right)$ for large k and upstream for small k (long wavelength).