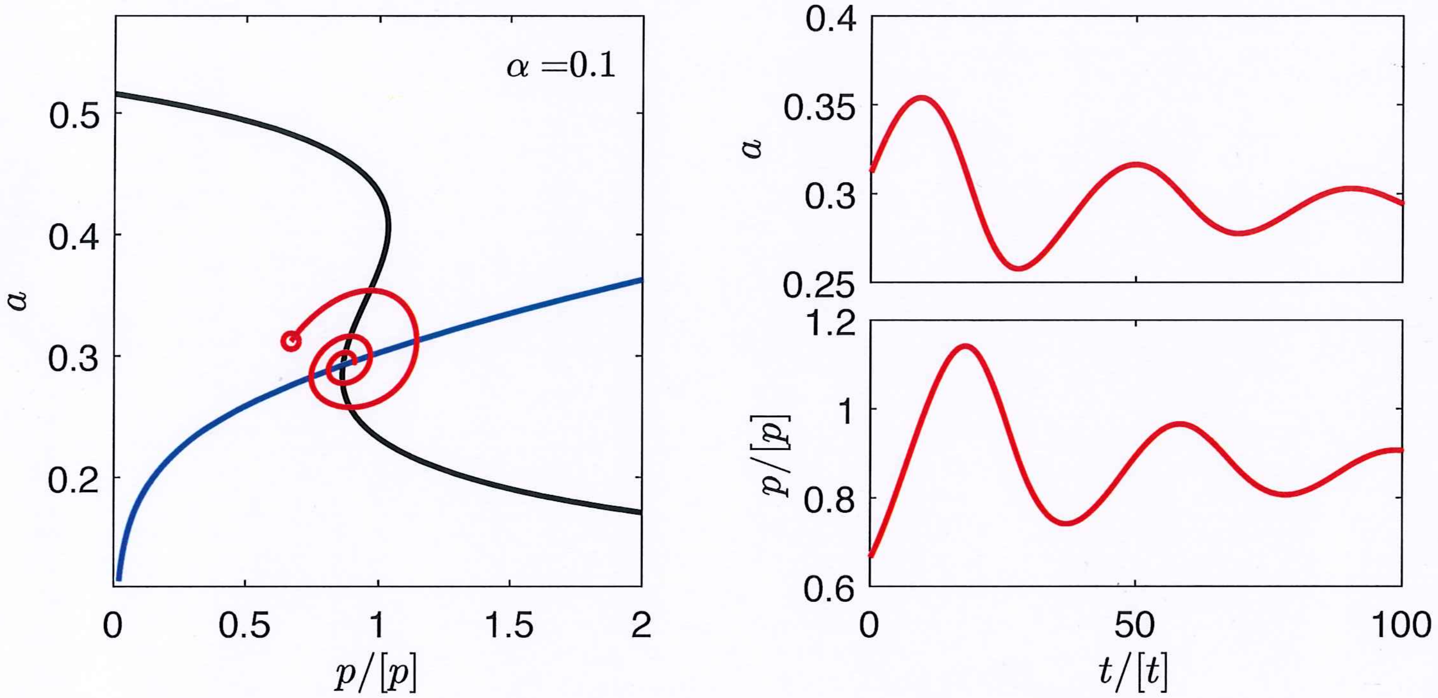
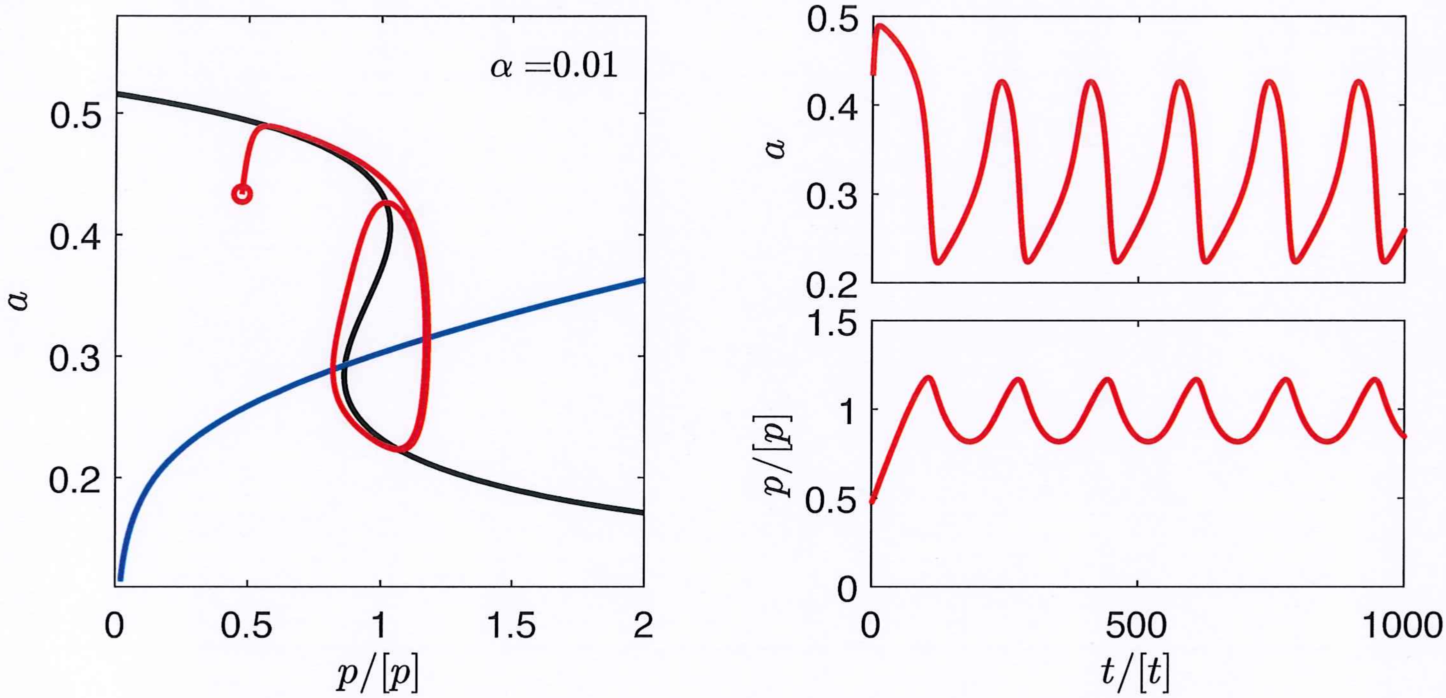
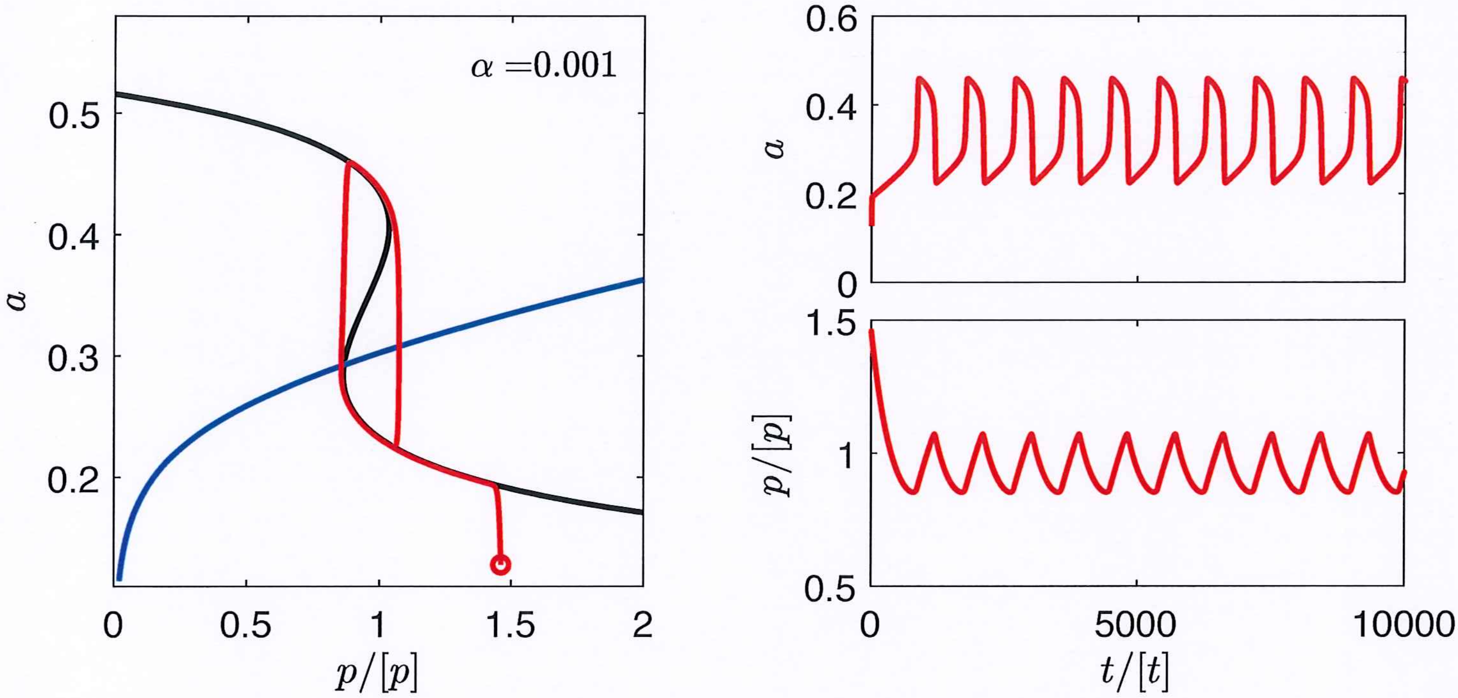
Extension Sheet.

1. Carbon cycles (from carbon-cycles.m).



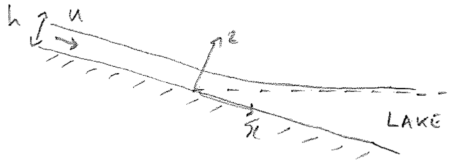


α =0.01
a
$p/[p]$
$t/[t]$
0
0.5
1
1.5
2
0.2
0.3
0.4
0.5
500
1000



α =0.001
0.5
0.4
0.3
0.2
a
0
0.5
1
1.5
2
$p/[p]$
0.6
0.4
0.2
0
a
1.5
1
0.5
$p/[p]$
0
5000
10000
$t/[t]$

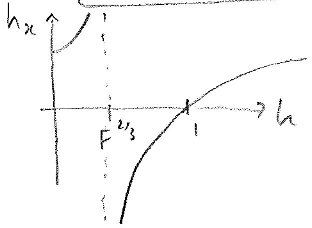
2. <u>River Mouth</u> $\quad h_t + (hu)_x = 0 \qquad F^2(u_t + uu_x) = -h_x + 1 - \frac{u^2}{h}$



The lake level must be horizontal far from the river mouth, and since the coordinates are angled relative to the horizontal (evidenced by the 1 on the right hand side of the momentum equations, which is the downslope component of gravity) we must have $h \sim x$ as $x \to \infty$.

In steady state we have $uh = 1$ (from upstream conditions) and

$$F^2 uu_x = -h_x + 1 - \frac{u^2}{h} \Rightarrow \left(1 - \frac{F^2}{h^3}\right)h_x = 1 - \frac{1}{h^3} \Rightarrow \boxed{h_x = \frac{h^3 - 1}{h^3 - F^2}}$$



If <u>$F<1$</u>, there is a solution with h varying monotonically from 1 to $h \sim x$ as x goes from $-\infty$ to ∞.

rearranging $\Rightarrow \left(1 + \frac{1-F^2}{h^3-1}\right)h_x = 1$

$$\frac{1}{h^3-1} = \frac{1}{3}\left[\frac{1}{h-1} - \frac{h+2}{h^2+h+1}\right]$$

$$= \frac{1}{3}\frac{1}{h-1} - \frac{1}{6}\frac{2h+1}{h^2+h+1} - \frac{1}{2}\frac{1}{(h+\frac{1}{2})^2+\frac{3}{4}}$$

$$\Rightarrow h + (1-F^2)\left[\frac{1}{3}\ln(h-1) - \frac{1}{6}\ln(h^2+h+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right] = x + \text{const.}$$

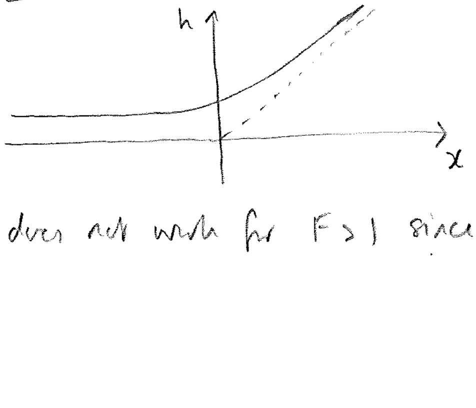
$$\left[\int \frac{dh}{(h+\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right]$$

i.e. $$h + \frac{(1-F^2)}{3}\ln\left(\frac{h-1}{(h^2+h+1)^{1/2}}\right) - \frac{1-F^2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right) = x + \text{const.}$$

This automatically satisfies $h \to 1$ as $x \to -\infty$. For $x \to \infty$, $h \sim x$ gives

$$x + 0 - \frac{1-F^2}{\sqrt{3}}\frac{\pi}{2} = x + \text{const.}, \quad \text{so const} = -\frac{\pi}{2\sqrt{3}}(1-F^2).$$

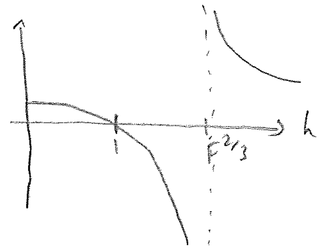
$$\Rightarrow \boxed{x = h + (1-F^2)\left[\frac{1}{3}\ln\left(\frac{h-1}{(h^2+h+1)^{1/2}}\right) + \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(h+\frac{1}{2}\right)\right)\right)\right]}$$



[It is easiest to draw this sketch by inspection of the ODE (see above) rather than using this ugly formula]

This solution does not work for $F>1$ since it cannot satisfy $h \to 1$ as $x \to -\infty$

For F larger than 1 the ODE $h_x = \dfrac{h^3-1}{h^3-F^2}$ looks like:



The state $h=1$ is a stable fixed pt and it is not possible to have a solution that departs from it as x increases. The only way to connect the upstream condition $h=1$ with the downstream behaviour $h \sim x$ is to have a shock. If the shock is at $x = x_s$, we have $h=1$ for $x < x_s$, and h given by the solution above for $x > x_s$.

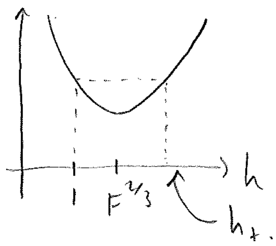
We have already invoked conservation of mass ($hu=1$); we must also conserve momentum across the shock. The momentum equation comes from combining mass and force balance eqns as

$$F^2\left((hu)_t + (hu^2)_x\right) = -hh_x + h - u^2$$

ie. $$\boxed{F^2(hu)_t + \left(F^2hu^2 + \tfrac12 h^2\right)_x = h - u^2}$$

Since the shock is steady, the shock condition is simply $\left[F^2 \underbrace{hu^2}_{1/h} + \tfrac12 h^2\right]_-^+ = 0.$

ie. $$\left.\frac{F^2}{h} + \frac12 h^2\right|_+ = F^2 + \frac12$$



$$\Rightarrow F^2(1-h) = \tfrac12(1-h^2)h \quad \Big\downarrow h \neq 1$$

$$\Rightarrow F^2 = \tfrac12 h(1+h)$$

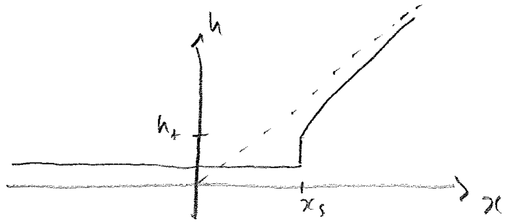
$$\Rightarrow \left(h+\tfrac12\right)^2 = \tfrac14 + 2F^2 \Rightarrow \boxed{h_+ = \tfrac12\left[-1 + (1+8F^2)^{1/2}\right]}$$

so
$$h_+ + \tfrac12 = \tfrac12(1+8F^2)^{1/2}$$
$$h_+ - 1 = -\tfrac32 + \tfrac12(1+8F^2)^{1/2}$$
$$h_+^2 + h_+ + 1 = 1 + 2F^2$$

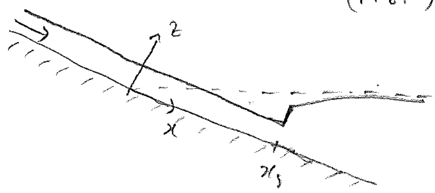
Since we know h_+ in terms of x_s (the solution of the ODE found above), this tells us the position of the shock, ie

$$x_s = h_+ - (F^2-1)\left[\frac13 \ln\left(\frac{h-1}{(h^2+h+1)^{1/2}}\right) + \frac{1}{\sqrt3}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt3}\left(h+\frac12\right)\right)\right)\right]$$

$$= -\frac12 + \frac12(1+8F^2)^{1/2} - (F^2-1)\left[\frac13 \ln\left(\frac{-3+(1+8F^2)^{1/2}}{2(1+2F^2)^{1/2}}\right) + \frac{1}{\sqrt3}\underbrace{\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\sqrt3}(1+8F^2)^{1/2}\right)\right)}_{\tan^{-1}\left(\frac{\sqrt3}{(1+8F^2)^{1/2}}\right)}\right].$$



ie.

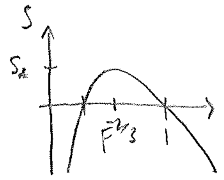


3. Antidunes $\quad (hu)_x = 0 \quad \left(\tfrac{1}{2}F^2u^2 + s + h\right)_x = 0$

$$s_t + q_x = 0 \qquad q_x = q^*(u) - q$$

- Integrating the water mass & momentum eqns and using that $s=0, h=u=1$ at some point

$\Rightarrow \quad hu = 1, \quad \tfrac{1}{2}F^2u^2 + s + h = \tfrac{1}{2}F^2 + 1$



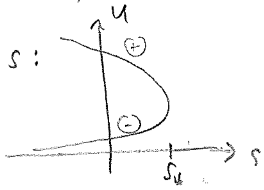
(drawn for the case $F>1$)

Eliminate $h = \frac{1}{u} \Rightarrow \quad \boxed{s = \tfrac{1}{2}F^2(1-u^2) + 1 - \tfrac{1}{u}}$

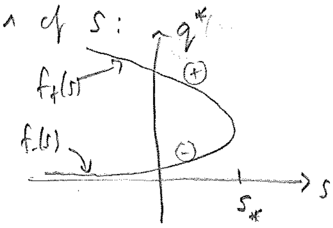
$\frac{\partial s}{\partial u} = -F^2u + \frac{1}{u^2} = 0$ at $u = F^{-2/3}$, where $s = \tfrac{1}{2}F^2\left(1 - \frac{1}{F^{4/3}}\right) + 1 - F^{2/3}$

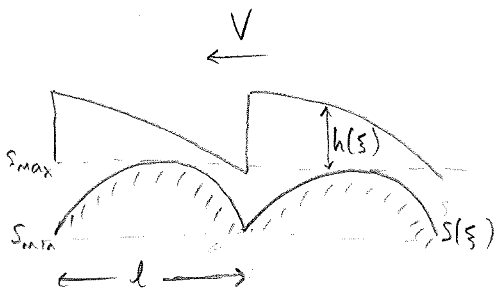
$$= 1 + \tfrac{1}{2}F^2 - \tfrac{3}{2}F^{2/3} =: s_*$$

- Since $q^*(u)$ is a monotonic function and u is the multivalued function of s:



q^* can be interpreted as a similar multivalued function of s:

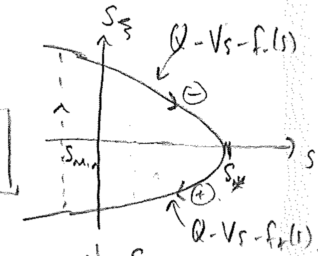




- Writing $\xi = x + Vt$ and $s = s(\xi)$, the equations transform according to $\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial \xi}, \ \frac{\partial}{\partial t} \mapsto V\frac{\partial}{\partial \xi}$

So $\quad Vs_\xi + q_\xi = 0 \qquad q_\xi = q^* - q$

$\Rightarrow \boxed{Vs + q = Q}$, constant, and $\quad \boxed{Vs_\xi = q - q^* = Q - Vs - f_\pm(s)}$



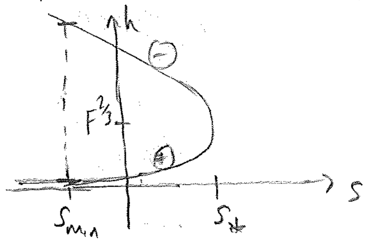
- s must increase from s_{min} to s_{max} and back again smoothly, so $s_\xi = 0$ at s_{max} and we need $s_\xi > 0$ from s_{min} to s_{max} and $s_\xi < 0$ from s_{max} to s_{min}. This increase and decrease must occur on the branches $f_-(s)$ and $f_+(s)$ respectively, and for s_ξ to be continuous the transition (at s_{max}) must occur where $f_+(s) = f_-(s)$, i.e. $s_{max} = s_*$

Since $s_\xi = 0$ at $s_{max} = s_*$, we need $\boxed{Q = Vs_* + f_\pm(s_*)}$.

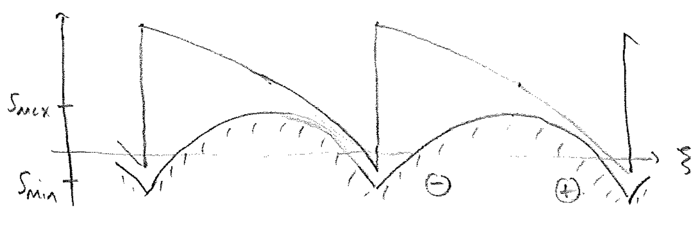
- Wavelength from summing the two sections $[s_{min}, s_{max}]$ and $[s_{max}, s_{min}]$

$$l = \int_{s_{min}}^{s_{max}} \frac{ds}{s_\xi} + \int_{s_{max}}^{s_{min}} \frac{ds}{s_\xi} = \int_{s_{min}}^{s_{max}} \left\{ \frac{V}{Q - Vs - f_-(s)} - \frac{V}{Q - Vs - f_+(s)} \right\} ds$$

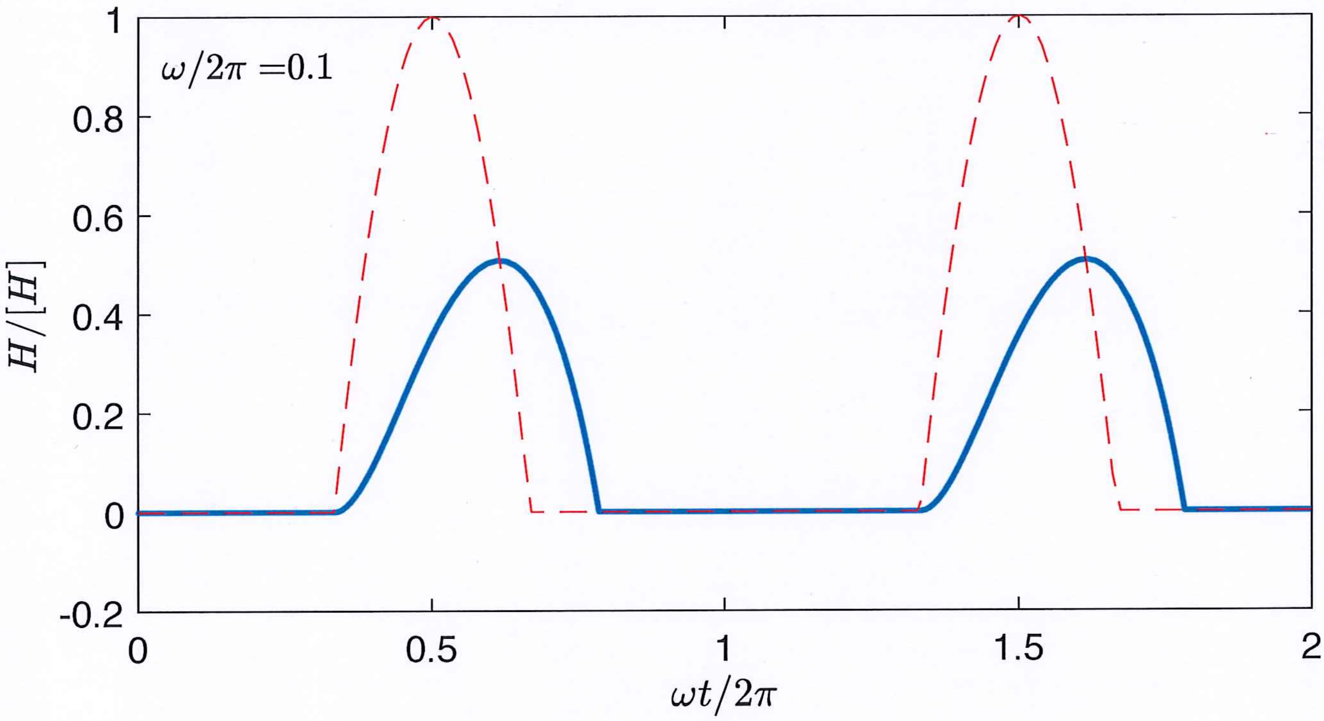
Given the multivalued relationship between u and s, we have a similar relationship between $h = \frac{1}{u}$ and s:

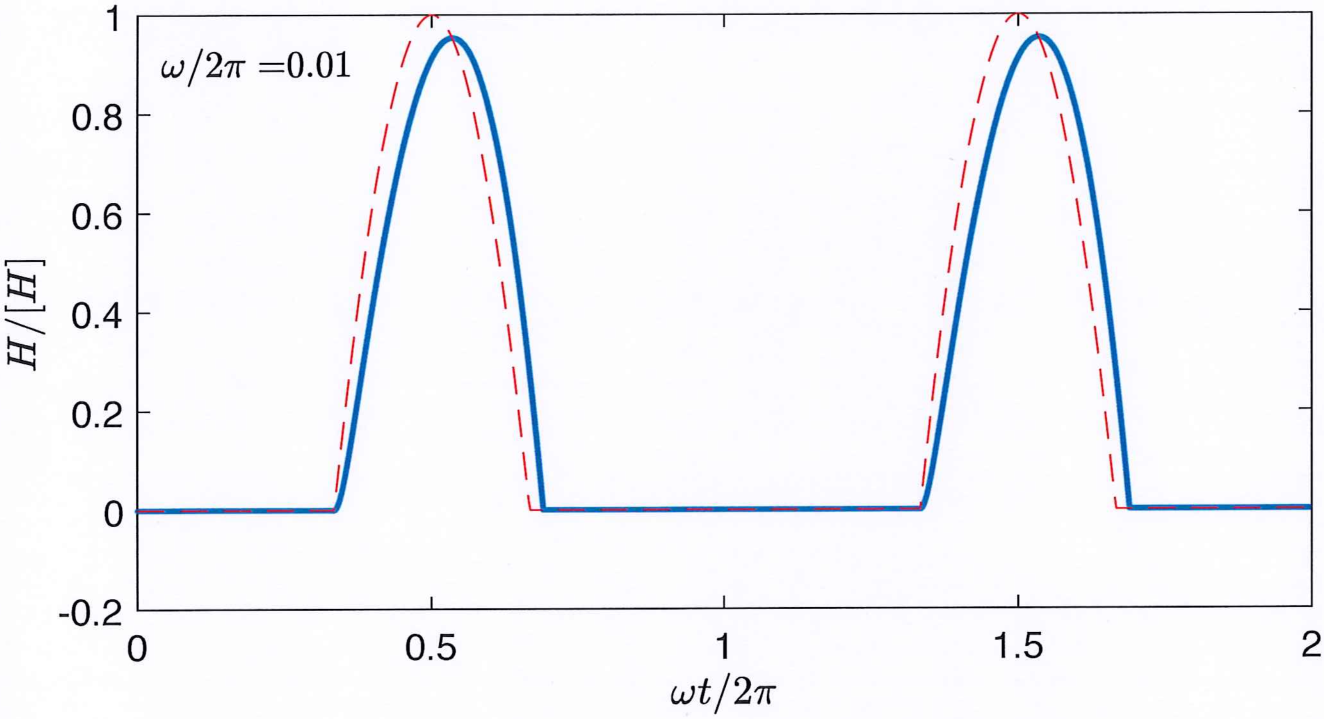


When we jump from the $\oplus$ branch to the $\ominus$ branch at s_{min}, there is a sudden increase in h (a <u>hydraulic jump</u>). The transition at s_* occurs where $h = F^{2/3}$ – this is called a <u>hydraulic control point</u>, where the flow transitions from subcritical ($\ominus$) to supercritical ($\oplus$). It is common to see this transition at the crest of a weir.

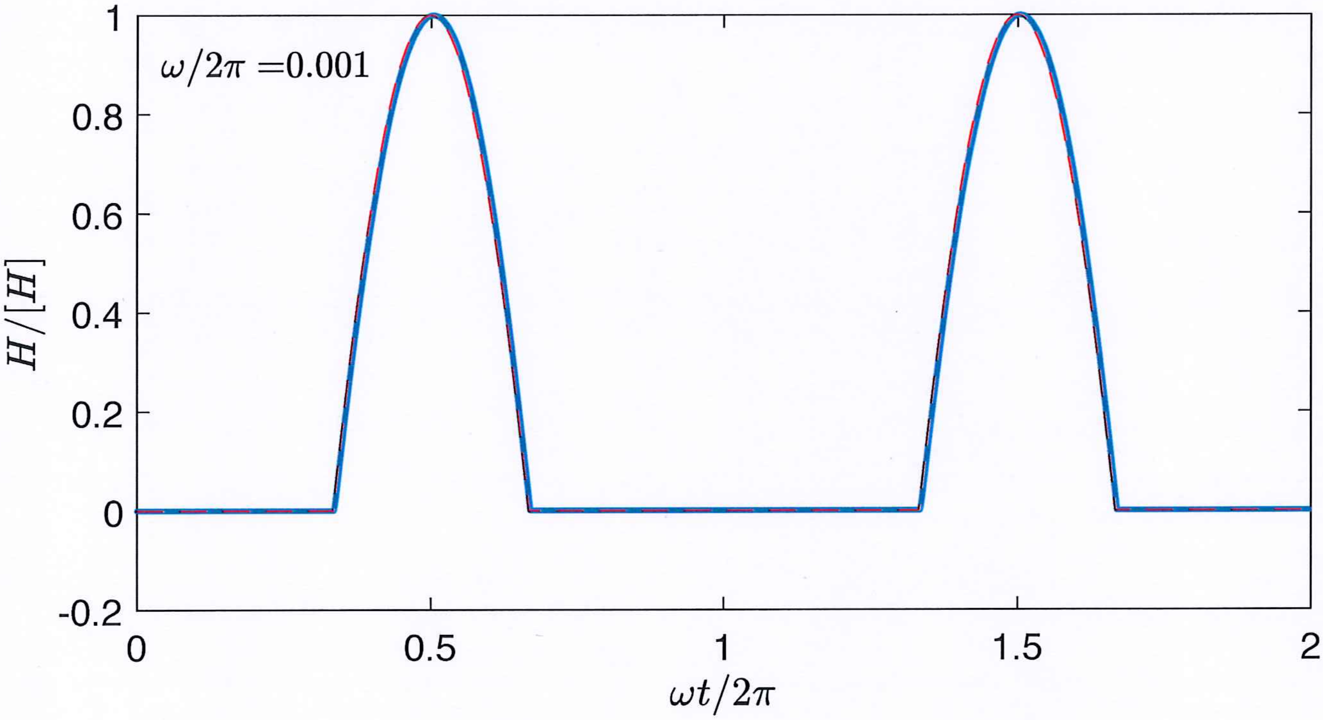


4. Sea ice (see sea_ice.m).





$\omega/2\pi$ =0.01
$H/[H]$
$\omega t/2\pi$
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2



$\omega/2\pi$ =0.001
$H/[H]$
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
$\omega t/2\pi$