Problem sheet 0

[*These questions are intended as introductory/background questions that give a taste of the sorts of models we consider during this course. They can be done before the start of term and solutions are available online.*]

1. **Planetary temperatures** A simple energy balance model for the temperature T of a planetary atmosphere is

$$\rho c_p d \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4} Q(1-a) - \sigma T^4,$$

where ρ, c_p and d are the average density, specific heat capacity, and depth of the atmosphere, Q is the solar radiation, a is the planetary albedo, all assumed constant, and $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is the Stefan-Boltzmann constant.

Find the steady state temperature and show that it is a stable equilibrium.

Calculate the steady state temperatures and compare with measured surface temperatures T_m for Earth ($Q = 1370$ W m^{-2}, $a = 0.3$, $T_m = 290$ K), Venus (0.72 a.u., $a = 0.77$, $T_m = 740$ K), Mars (1.52 a.u., $a = 0.15$, $T_m = 220$ K), and Jupiter (5.2 a.u., $a = 0.58$, $T_m = 130$ K). 1 a.u. is the distance from the sun to the Earth.

[*Hint: think about how you expect Q to vary with relative distance from the sun. This problem ignores the greenhouse effect, which we will include in lectures.*]

2. **Dam** A simple dimensionless model for the depth of water in a river $h(x,t)$ is

$$\frac{\partial h}{\partial t} + h^m \frac{\partial h}{\partial x} = 0,$$

where x is distance along the centreline of the river and $m > 0$ is a constant.

A dam at $x = 0$ controls the flow in the river downstream. Suppose that at time $t = 0$ the outflow from the dam is suddenly decreased such that $h(0,t)$ decreases from h_1 (at which it has been held for a long time) to h_2. Use the method of characteristics to solve for the water depth, as a function of time, at a distance L downstream.

3. **Seasonal wave** Ground temperature is governed by the diffusion equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2},$$

with boundary conditions $\partial T/\partial z \to 0$ as $z \to \infty$ and $T = T_0 - \Delta T \cos \omega t$ at $z = 0$ (the z coordinate points vertically downwards from the surface).

Non-dimensionalise the model, and solve the resulting dimensionless equation and boundary conditions. Using values $\rho = 1600$ kg m^{-3}, $c = 800$ J kg^{-1} K^{-1}, and $k = 1$ W m^{-1} K^{-1}, estimate the depth over which (a) daily, and (b) annual temperature variations decay.

4. **Stefan-Boltzmann law** Given the Planck function $B_\nu = 2h\nu^3/c^2(e^{h\nu/kT} - 1)$ for the emission of radiation with frequency ν, show that the total emission is

$$B = \int_0^\infty B_\nu \, \mathrm{d}\nu = \frac{\sigma T^4}{\pi},$$

where the Stefan-Boltzmann constant is

$$\sigma = \frac{2\pi k^4}{h^3 c^2} \int_0^\infty \frac{u^3 \, \mathrm{d}u}{e^u - 1} = \frac{2\pi^5 k^4}{15 h^3 c^2}.$$

[*Hint: the last integral can be evaluated by making use of the identity* $\sum_1^\infty 1/n^4 = \pi^4/90$.]