Topics in fluid mechanics

PROBLEM SHEET 4.

1. Derive a reference state for a dry atmosphere (no condensation) by using the equation of state
$$p = \frac{\rho R T}{M_a},$$
the hydrostatic pressure
$$\frac{\partial p}{\partial z} = -\rho g,$$
and the dry adiabatic temperature equation
$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = 0.$$
Show that
$$\bar{T} = T_0 - \frac{gz}{c_p}, \quad \bar{p} = p_0 p^*(z),$$
where
$$p^*(z) = \left(1 - \frac{gz}{c_p T_0}\right)^{M_a c_p/R}.$$
Use the typical values $c_p T_0/g \approx 29$ km, $M_a c_p/R \approx 3.4$, to show that the pressure can be adequately represented by
$$\bar{p} = p_0 \exp(-z/H),$$
where here the scale height is defined as
$$H = \frac{RT_0}{M_a g} \approx 8.4 \ \text{ km}.$$
(A slightly better numerical approximation near the tropopause is obtained if the scale height is chosen as 7 km.)

2. The mass and momentum equations for atmospheric motion in the rotating frame of the Earth can be written in the form
$$\rho_t + \boldsymbol{\nabla}.\,[\rho \mathbf{u}] = 0,$$
$$\rho\left[\frac{d\mathbf{u}}{dt} + 2\boldsymbol{\Omega}\times\mathbf{u}\right] = -\boldsymbol{\nabla} p - \rho g \hat{\mathbf{k}},$$
where (x, y, z) are local Cartesian coordinates at latitude $\lambda = \lambda_0$. What is the magnitude of Ω?

Scale the variables by writing
$$x, y \sim l, \quad z \sim h, \quad u, v \sim U, \quad w \sim \delta U, \quad t \sim \frac{l}{U},$$

$$\rho \sim \rho_0, \quad T \sim T_0, \quad p = p_0\bar{p}(z) + 2\rho_0\Omega U l \sin\lambda_0\, P,$$

where

$$\delta = \frac{h}{l}, \quad p_0 = \rho_0 g h = \frac{\rho_0 R T_0}{M},$$

and show that the horizontal components take the form

$$\varepsilon\frac{du}{dt} - fv = -\frac{1}{\rho}P_x,$$

$$\varepsilon\frac{dv}{dt} + fu = -\frac{1}{\rho}P_y,$$

where

$$f = \frac{\sin\lambda}{\sin\lambda_0},$$

and give the definition of the Rossby number ε. Show that in a linear approximation,

$$f \approx 1 + \varepsilon\beta y,$$

where

$$\beta = \frac{l}{R_E}\frac{\cot\lambda_0}{\varepsilon} = O(1),$$

and R_E is Earth's radius.

The dimensionless pressure $\Pi = p/p_0$, density ρ, temperature T and potential temperature θ in the atmosphere satisfy the relations

$$\rho = \frac{\Pi}{T}, \quad T = \theta\Pi^\alpha, \quad -\frac{\partial\Pi}{\partial z} = \rho,$$

where $\alpha = \dfrac{R}{M_a c_p}$ is constant. Assuming that

$$\Pi = \bar{p} + \varepsilon^2 P, \quad \theta = \bar{\theta} + \varepsilon^2\Theta,$$

and that $\varepsilon \ll 1$, deduce that $\rho \approx \bar{\rho}(z)$, and thence that

$$w = O(\varepsilon), \quad \bar{\rho}u \approx -P_y, \quad \bar{\rho}v \approx P_x.$$

Show also that consistency between the two forms of scaled pressure requires the definition of the velocity scale to be

$$U = \frac{8(\Omega l \sin\lambda_0)^3}{gh},$$

and determine this value, if $l = 1{,}000$ km, $\lambda_0 = 45°$, $g = 9.8$ m s^{-2}, $h = 8$ km.

Show that

$$\Theta \approx \bar{\theta}^2\frac{\partial}{\partial z}\left[\frac{P}{\bar{p}^{1-\alpha}}\right],$$

and by defining a stream function via $P = \bar{\rho}\psi$ and assuming that $\bar{\theta} \approx 1$, deduce that $\Theta \approx \psi_z$, and hence deduce the *thermal wind equations*:

$$\frac{\partial u}{\partial z} = -\frac{\partial \Theta}{\partial y}, \quad \frac{\partial v}{\partial z} = \frac{\partial \Theta}{\partial x}.$$

3. The quasi-geostrophic potential vorticity equation is

$$\frac{d}{dt}\left[\nabla^2\psi + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\frac{\bar{\rho}}{S}\frac{\partial \psi}{\partial z}\right)\right] + \beta\psi_x = \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\frac{\bar{\rho}H}{S}\right),$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, and $\bar{\rho}$, S and H are functions of z, the first two being positive. The horizontal material derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \quad u = -\psi_y, \quad v = \psi_x.$$

In the Eady model of baroclinic instability, solutions to the QGPVE are sought in a channel $0 < y < 1$, $0 < z < 1$, with boundary conditions

$$\frac{d}{dt}\psi_z = 0 \quad \text{at} \quad z = 0, 1, \qquad \psi_x = 0 \quad \text{at} \quad y = 0, 1,$$

and it is supposed that $\bar{\rho}$ and S are constant, and $\beta = H = 0$. Show that a particular solution is the zonal flow $\psi = -yz$, and describe its velocity field. By considering the thermal wind equations, explain why this is a meaningful solution.

By writing $\psi = -yz + \Psi$ and linearising the equations, derive an equation for Ψ, and show that it has solutions

$$\Psi = A(z)e^{ik(x-ct)}\sin n\pi y,$$

providing

$$(z-c)(A'' - \mu^2 A) = 0,$$
$$(z-c)A' = A \quad \text{on} \quad z = 0, 1,$$

where you should define μ.

Using the fact that $x\delta(x) = 0$, show that if $0 < c < 1$, the solution can be found as a Green's function for the equation $A'' - \mu^2 A = 0$.

Give a criterion for instability, and show that for the normal mode solutions in which A is analytic,

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left\{\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right\}^{1/2},$$

and hence show that the zonal flow is unstable if $\mu < \mu_c$, where

$$\frac{\mu}{2} = \coth\frac{\mu}{2},$$

and calculate this value. Deduce that the flow is unstable for $S < S_c$, and calculate S_c.

4. A basic two fluid model of two-phase flow is given by the equations

$$(\alpha\rho_g)_t + (\alpha\rho_g v)_z = \Gamma,$$
$$\{\rho_l(1-\alpha)\}_t + \{\rho_l(1-\alpha)u\}_z = -\Gamma,$$
$$\rho_g[v_t + vv_z] = -p_z - M,$$
$$\rho_l[u_t + D_l uu_z] = -p_z + M,$$

where α is void fraction, u and v are liquid and gas phase velocities, p is pressure, and ρ_g and ρ_l are gas and liquid densities; the constant $D_l > 1$ is a profile coefficient, and Γ and M are interfacial source and drag terms, which are prescribed algebraic functions of the variables.

Explain how to find the characteristics of this system when written in the form

$$A\boldsymbol{\psi}_t + B\boldsymbol{\psi}_z = \mathbf{c}.$$

(i) Assuming ρ_g and ρ_l are constant and $\rho_g \ll \rho_l$, show that the characteristics are generally real.

(ii) If

$$\frac{d\rho_g}{dp} = \frac{1}{c_g^2}, \quad \frac{d\rho_l}{dp} = \frac{1}{c_l^2},$$

calculate approximate values of the characteristics if $u \sim v \ll c_l \sim c_g$ and $\rho_g \ll \rho_l$, and comment on the physical significance of these.

5. The energy equation for a one-dimensional two-phase flow in a tube is given by

$$\Gamma L + \alpha\rho_g c_{pg}(T_t + vT_z) + (1-\alpha)\rho_l c_{pl}(T_t + uT_z) - \{(\alpha p_g)_t + (\alpha p_g v)_z\}$$
$$-[\{(1-\alpha)p_l\}_t + \{(1-\alpha)p_l u\}_z] = Q,$$

where

$$\Gamma = (\alpha\rho_g)_t + (\alpha\rho_g v)_z = -[\{(1-\alpha)\rho_l\}_t + \{(1-\alpha)\rho_l u\}_z],$$

and the temperatures of the two phases are assumed equal, and denoted by T.

The enthalpy of each phase satisfies $dh_k = c_{pk}\,dT$, and is related to the internal energy e_k by

$$h_k = e_k + \frac{p_k}{\rho_k};$$

$L = h_g - h_l$ is the latent heat. Deduce that the energy equation can be written in the form

$$(\alpha\rho_g e_g)_t + (\alpha\rho_g e_g v)_z + [(1-\alpha)\rho_l e_l]_t + [(1-\alpha)\rho_l e_l u]_z = Q.$$

Define the mixture density by

$$\rho = \rho_l(1-\alpha) + \rho_g\alpha,$$

the mixture pressure by

$$p = (1-\alpha)p_l + \alpha p_g,$$

the mixture internal energy by

$$\rho e = \alpha \rho_g e_g + (1-\alpha)\rho_l e_l,$$

and the mixture enthalpy by

$$h = e + \frac{p}{\rho};$$

deduce that

$$\rho h = \alpha \rho_g h_g + (1-\alpha)\rho_l h_l.$$

If the flow is homogeneous, deduce that

$$\rho \frac{de}{dt} = 0,$$

where $\frac{d}{dt}$ is the material derivative, and if the pressure drop along the tube $\Delta p \ll \rho_g L$, show that $h \approx e$, and deduce that

$$\frac{\partial u}{\partial z} = \frac{(\rho_l - \rho_g)Q}{\rho_g \rho_l L}.$$

6. An approximate homogeneous two-phase model for density wave oscillations in a pipe of length l is given by

$$\rho_t + u\rho_z = -u_z \rho,$$

$$\rho(u_t + uu_z) = -p_z - \rho g - \frac{4f\rho u^2}{d},$$

$$\rho(h_t + uh_x) = Q,$$

where Q is constant, and

$$h \approx h^* + \frac{\rho_g L}{\rho}$$

in the two-phase region; h^*, L and Q are constants, ρ_g and ρ_l are (constant) gas and liquid densities, h is enthalpy, and ρ, p and u are mixture density, pressure and velocity. For $h < h_{\text{sat}}$, the saturation enthalpy, only liquid is present, $\rho = \rho_l$, and the above relation for h is irrelevant.

Boundary conditions for the flow are that

$$h = h_0 < h_{\text{sat}}, \quad u = U(t) \quad \text{at} \quad z = 0,$$

$$h = h_{\text{sat}} \quad \text{on} \quad z = r(t),$$

where the unknown boiling boundary $r(t)$ is to be determined, and the pressure drop along the pipe, Δp, is prescribed.

Show that
$$r(t) = \int_{t-\tau}^{t} U(s)\,ds,$$
and give the definition of τ.

Non-dimensionalise the two-phase model by scaling
$$\rho \sim \rho_l, \quad z, r \sim l, \quad t \sim \tau, \quad u, U \sim u_0,$$
and show that the two-phase velocity and density satisfy
$$u = U + \frac{z-r}{\varepsilon}, \quad z = r + \varepsilon \int_0^{-\ln \rho} U_1(t - \varepsilon\xi) e^{\xi}\,d\xi, \quad r = \int_{t-1}^{t} U(s)\,ds,$$
where $U_1(t) = U(t-1)$, and give the definition of ε.

Show that the pressure drop in the single phase region is
$$\Delta p_{sp} = [\Delta p_i \dot{U} + \Delta p_g + \Delta p_f U^2] r,$$
where
$$\Delta p_i = \rho_l u_0^2, \quad \Delta p_g = \rho_l g l, \quad \Delta p_f = \frac{4 f l \rho_l u_0^2}{d}, \quad u_0 = \frac{l}{\tau}.$$

Write down an integral expression for the two-phase pressure drop in the form
$$\Delta p_{tp} = \int_r^1 (\Delta p_i \Phi_i + \Delta p_g \Phi_g + \Delta p_f \Phi_f]\,dz,$$
where the functions Φ_k depend on u and ρ and their derivatives.

If $U = V$ in the steady state, explain why $0 < V < 1$. Write down an expression for Δp as a function of V. Show that if V is sufficiently close to one, Δp is an increasing function of V, but that if ε is sufficiently small, it is a decreasing function of V.

Now suppose that $\Delta p_i = \Delta p_g = 0$. To examine the stability of the steady state (denoted by a suffix zero for r, u and ρ), write
$$U = V + v, \quad r = r_0 + r_1, \quad u = u_0 + u_1, \quad \rho = \rho_0 + \rho_1,$$
and linearise the equations. Hence derive expressions for r_1, u_1 and ρ_1.

By taking $v = e^{\sigma t}$, derive an algebraic equation for σ from the condition that the perturbation to Δp is zero. If only the single phase pressure drop term is included, show that
$$\sigma = -\tfrac{1}{2}(1 - e^{-\sigma}),$$
and deduce that the steady state is stable.

If only the two-phase pressure drop is included, and ε is assumed to be small, show that
$$\sigma = \gamma(e^{\sigma} - 1), \quad \gamma = \frac{2V}{1-V},$$

and deduce that $\mathrm{Re}\,\sigma \to \infty$ as $\sigma \to \infty \in \mathbf{C}$, and thus that the model is ill-posed.

If both pressure drops are included (and the two-phase approximation for small ε is used), show that

$$\sigma = \frac{\gamma(1-e^{-\sigma})}{\delta + e^{-\sigma}}, \quad \delta = \frac{4\varepsilon V^2}{(1-V)^2},$$

and deduce that the model is ill-posed for $\delta < 1$.

Finally, if the inertial term in the single phase region (only) is included, show that

$$\nu\sigma^2 + \sigma(\delta + e^{-\sigma}) - \gamma(1-e^{-\sigma}) = 0, \quad \nu = \frac{2\varepsilon \Delta p_i}{(1-V)^2 \Delta p_f},$$

and deduce that the model is well-posed, but the steady state is unstable for small ε.