Topics in fluids Sheet 4 Answers

1. $$p = \frac{\rho R T}{M_a}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = 0 \Rightarrow \rho c_p \frac{\partial T}{\partial z} = \frac{\partial p}{\partial z} = -\rho g$$

$$\Rightarrow \frac{\partial T}{\partial z} = -\frac{g}{c_p}$$

$$\Rightarrow T = \bar{T} = T_0 - \frac{gz}{c_p}, \quad p = \bar{p} = p_0 p^*(z)$$

where $$p_0 p^{*\prime} = \frac{\partial p}{\partial z} = -\rho g = -\frac{M_a p g}{RT} = \frac{-M_a p_0 p^* g}{R\left(T_0 - \frac{gz}{c_p}\right)}$$

So $$\frac{p^{*\prime}}{p^*} = \frac{-M_a g}{RT_0\left(1 - \frac{gz}{c_p T_0}\right)}, \quad p^*(0) = 1$$

$$\Rightarrow \ln p^* = +\frac{M_a g}{RT_0} \cdot \frac{c_p T_0}{g} \ln\left[1 - \frac{gz}{c_p T_0}\right]$$

$$\Rightarrow p^* = \left(1 - \frac{gz}{c_p T_0}\right)^{M_a c_p / R}$$

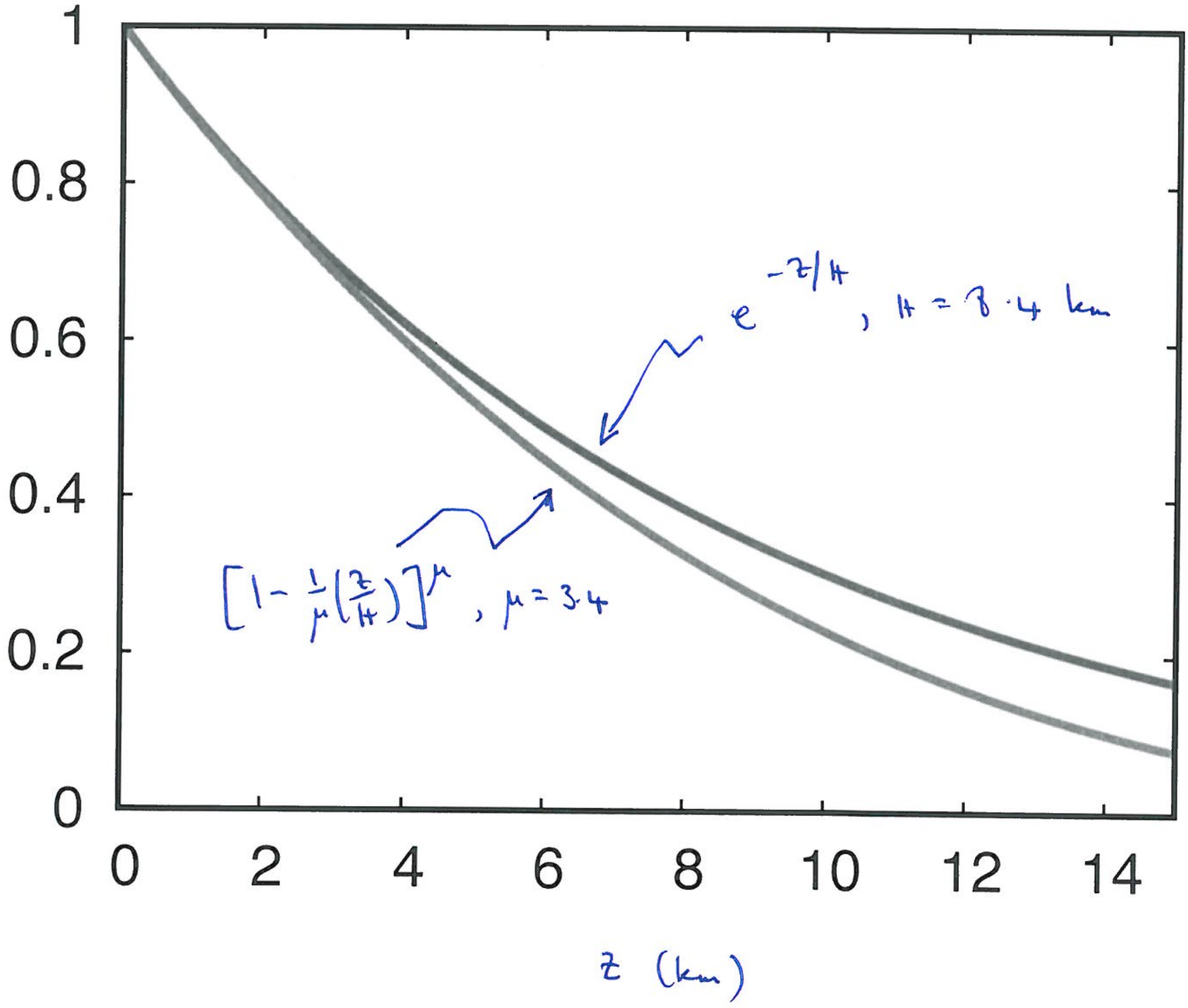
Define $H = \frac{RT_0}{M_a g}$, $$p^* = \left(1 - \frac{z}{H} \frac{RT_0}{M_a g} \cdot \frac{g}{c_p T_0}\right)^{M_a c_p / R}$$

$$= \left[1 - \frac{1}{\mu}\left(\frac{z}{H}\right)\right]^{\mu}, \quad \mu = \frac{M_a c_p}{R} = 3.4$$

So for $z \sim H$ & since $\mu \gg 1$, $p^* \approx \exp\left(-\frac{z}{H}\right)$

The approximation is not bad for $z \lesssim H$

(and is anyway inappropriate above the tropopause).



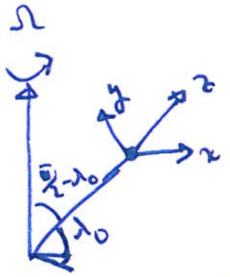
2)

$$\rho_t + \underline{\nabla}.(\rho \underline{u}) = 0$$

$$\rho\left[\frac{d\underline{u}}{dt} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}\right] = -\underline{\nabla}p - \rho g \hat{\underline{k}}$$

The magnitude of Ω is

$$2\pi\, d^{-1} = \frac{2\pi}{24\times3600}\, s^{-1} \simeq 0\cdot73\times10^{-4}\, s^{-1}$$



Note $\underline{\Omega} = (0, \Omega\cos\lambda, \Omega\sin\lambda)$

$$\text{so } \underline{\Omega}\times\underline{u} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ u & v & w \end{pmatrix} \approx (-\Omega v\sin\lambda, \Omega u\sin\lambda, -\Omega u\cos\lambda)$$

since $w \ll u, v$

Scaling the variables as prescribed in the question, we get

$$\rho_t + \underline{\nabla}.(\rho\underline{u}) = 0$$

$$\rho_0\rho\left[\frac{U^2}{l}\frac{du}{dt} - 2\Omega U\sin\lambda_0\, f v\right] = -\cancel{2}\rho_0\Omega U\sin\lambda_0\, P_x$$

& thus $\left(\div 2\rho_0\Omega U\sin\lambda_0)\rho\right]$

$$\frac{U^2}{l\,2\Omega U\sin\lambda_0}\frac{du}{dt} - f v = -\frac{1}{\rho}P_x \quad , \quad f = \frac{\sin\lambda}{\sin\lambda_0}$$

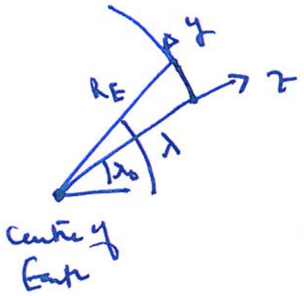
i.e

$$\underline{\varepsilon\frac{du}{dt} - f v = -\frac{1}{\rho}P_x}$$

$$\varepsilon = \frac{U}{2\Omega l\sin\lambda_0} \quad \text{Rossby number}$$

Similarly (the scales of the terms are the same)

$$\varepsilon \frac{dv}{dt} + f u = -\frac{1}{\rho} p_y$$



we have $y = R_E \sin(\lambda - \lambda_0)$

and for $y \ll R_E$, $\lambda \simeq \lambda_0$ & $y \approx R_E(\lambda - \lambda_0)$

thus $\lambda \simeq \lambda_0 + \frac{y}{R_E}$

& $f = \dfrac{\sin(\lambda_0 + \frac{y}{R_E})}{\sin\lambda_0} \simeq \dfrac{\sin\lambda_0 \cos\frac{y}{R_E} + \cos\lambda_0 \sin\frac{y}{R_E}}{\sin\lambda_0}$

$= 1 - \frac{y^2}{2R_E^2} \cdots + \cot\lambda_0 \frac{y}{R_E} \cdots$

& non-dimensionally $f \simeq 1 + \frac{l \cot\lambda_0}{R_E} y$

$= 1 + \varepsilon\beta y$,

$$\beta = \frac{l \cot\lambda_0}{\varepsilon R_E}$$

$$\rho = \frac{\Pi}{T}, \quad T = \theta\,\Pi^{\alpha}, \quad -\frac{\partial \Pi}{\partial z} = \rho$$

$$\Pi = \bar{p} + \varepsilon^2 p \qquad \theta = \bar{\theta} + \varepsilon^2 \Theta$$

$$\rho = \frac{\Pi^{1-\alpha}}{\theta} = \bar{\rho} + O(\varepsilon^2), \quad \bar{\rho} = \frac{\bar{p}^{1-\alpha}}{\bar{\theta}}$$

Now (mon-d) $\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0$

Thus $\bar{\rho}(u_x + v_y) + (\bar{\rho} w)_z = O(\varepsilon^2)$

But also
$$\underline{\begin{aligned} -\bar{\rho} v &= -P_x + O(\varepsilon) \\ \bar{\rho} u &= -P_y + O(\varepsilon) \end{aligned}}$$
from momentum equation

so $\bar{\rho}(u_x + v_y) = O(\varepsilon)$

$\Rightarrow \underline{w = O(\varepsilon)}$ (from mass continuity)

Now we took $p = p_0 \Pi = p_0 \bar{p} + \varepsilon^2 p_0 P$

but also $p = p_0 \bar{p} + 2\rho_0 \Omega l U \sin\lambda_0 P$

and thus $\varepsilon^2 p_0 = 2 p_0 \Omega l U \sin\lambda_0$

i.e. $\frac{\rho_0 g h U^2}{(2\Omega l \sin\lambda_0)^2} = \rho_0 U (2\Omega l \sin\lambda_0)$

(earlier we had $p_0 \sim \rho_0 g h$)

$$\text{so } \underline{U = \frac{8(\Omega l \sin\lambda_0)^2}{g h}}$$

$$\lambda_0 = 45^\circ \Rightarrow \sin\lambda_0 \simeq 0.7$$

$$U = \frac{8\left[0.73\times10^{-4}\ 10^{6}\ \mathrm{m}\ 0.7\ \mathrm{s}^{-1}\right]^3}{9.8\ 8\ 10^{3}\ \mathrm{m\,s^{-2}}\ \mathrm{m}}$$

$$= \frac{8\ (0.7\times0.73)^3\ 10^{6}}{9.8\ 8\ 10^{3}}\ \mathrm{m\,s^{-1}}$$

$$\sim 12\ \mathrm{m\,s^{-1}}$$

Note then

$$\varepsilon = \frac{12}{2\ 0.73\ 10^{-4}\ 0.7\ 10^{6}} \sim 0.12$$

Next we have

$$-\frac{\partial\Pi}{\partial z} = \rho = \frac{\Pi^{1-\alpha}}{\theta}, \quad \Pi = \bar{p} + \varepsilon^2 p, \quad \theta = \bar{\theta} + \varepsilon^2\Theta$$

So

$$-\frac{\partial\bar{p}}{\partial z} - \varepsilon^2\frac{\partial p}{\partial z} = \frac{(\bar{p}+\varepsilon^2 p)^{1-\alpha}}{\bar{\theta}+\varepsilon^2\Theta} = \frac{\bar{p}^{1-\alpha}}{\bar{\theta}}\left[1+\varepsilon^2(1-\alpha)\frac{p}{\bar{p}}\cdots\right]\left[1-\frac{\varepsilon^2\Theta}{\bar{\theta}}\cdots\right]$$

$$= \frac{\bar{p}^{1-\alpha}}{\bar{\theta}} + \varepsilon^2\left\{\frac{(1-\alpha)p}{\bar{\theta}\,\bar{p}^{\alpha}} - \frac{\bar{p}^{1-\alpha}\Theta}{\bar{\theta}^2}\right\}\cdots$$

thus equating terms of $O(\varepsilon^2)$

$$\frac{\bar{p}^{1-\alpha}\Theta}{\bar{\theta}^2} = \frac{(1-\alpha)p}{\bar{\theta}\,\bar{p}^{\alpha}} + \frac{\partial p}{\partial z} \qquad \left(\text{as well as } \frac{\partial\bar{p}}{\partial z} = -\frac{\bar{p}^{1-\alpha}}{\bar{\theta}}\right)$$

So

$$\Theta = \bar{\theta}^2\left[\frac{(1-\alpha)p}{\bar{\theta}\,\bar{p}} + \frac{1}{\bar{p}^{1-\alpha}}\frac{\partial p}{\partial z}\right]$$

Now note

$$\frac{\partial}{\partial z}\frac{1}{\bar{p}^{1-\alpha}} = \frac{-(1-\alpha)}{\bar{p}^{2-\alpha}}\frac{\partial\bar{p}}{\partial z} = \frac{(1-\alpha)}{\bar{p}^{2-\alpha}}\frac{\bar{p}^{1-\alpha}}{\bar{\theta}} = \frac{(1-\alpha)}{\bar{\theta}\,\bar{p}}$$

Therefore $\Theta = \bar{\theta}^{2}\left[\rho \frac{\partial}{\partial z} \frac{1}{\bar{p}^{1-\alpha}} + \frac{1}{\bar{p}^{1-\alpha}} \frac{\partial \rho}{\partial z} \right]$

$$= \bar{\theta}^{2} \frac{\partial}{\partial z}\left[\frac{\rho}{\bar{p}^{1-\alpha}} \right]$$

Define $\rho = \bar{\rho}\,\psi$ and assume $\bar{\theta} \approx 1$

Then (geostrophy) $v \approx \frac{1}{\bar{\rho}} \rho_x$, $u \approx -\frac{1}{\bar{\rho}} \rho_y$
see p3

so $u \approx -\psi_y$, $v \approx \psi_x$

and $\Theta = \bar{\theta}^{2} \frac{\partial}{\partial z}\left[\frac{\bar{\rho}\,\psi}{\bar{p}^{1-\alpha}} \right] = \bar{\theta}^{-2} \frac{\partial}{\partial z}\left[\frac{\psi}{\bar{\theta}} \right]$ $\left(\bar{\rho} \approx \frac{\bar{p}^{1-\alpha}}{\bar{\theta}} \right)$

$\Rightarrow \Theta \approx \psi_z$

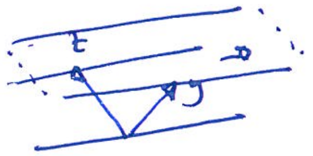
thus $\frac{\partial u}{\partial z} \approx -\frac{\partial \Theta}{\partial y}$, $\frac{\partial v}{\partial z} \approx \frac{\partial \Theta}{\partial x}$

3/ with $\bar{\rho}$ & S constant, $\beta = k = 0$,

$$\left[\frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right]\left[D^2\psi + \frac{1}{S}\psi_{zz}\right] = 0 \qquad \left(D^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

& $(\partial_t - \psi_y \partial_x + \psi_x \partial_y)\psi_z = 0$ at $z = 0, 1$

$\psi_x = 0$ at $y = 0, 1$



(flow in an unconfined channel)

$\psi = -yz$ is a particular solution satisfying the bc's + eq.

with $\Theta = \psi_z = -y$

& $\frac{\partial u}{\partial z} = -\Theta_y = 1$, $\frac{\partial v}{\partial z} = \Theta_x = 0$ (or just $u = -\psi_y = z$, $v = \psi_x = 0$)

this corresponds to uniform zonal shear flow.

Put $\psi = -yz + \Psi$, $\psi_y = -z + \Psi_y$, $\psi_x = \Psi_x$, $\psi_z = -y + \Psi_z$

$$D^2\psi + \frac{1}{S}\psi_{zz} = D^2\Psi + \frac{1}{S}\Psi_{zz} \ll 1$$

So linearised QGPVE is

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\left(D^2\Psi + \frac{1}{S}\Psi_{zz}\right)$$

& the linearised bc's are $\Psi_x = 0$ at $y = 0, 1$

& $-\Psi_x + \left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\Psi_z = 0$ on $z = 0, 1$.

(from $\uparrow$)

$$\left[\frac{\partial}{\partial t} - (-z + \Psi_y)\frac{\partial}{\partial x} + \Psi_x \frac{\partial}{\partial y}\right][-y + \Psi_z]$$

There are normal mode solutions

$$\Psi = A(z)e^{ik(x-ct)}\sin n\pi y \qquad \text{which satisfy the } y=0,1 \text{ b.c's}$$

[Note $\mathrm{Re}[-ikc] = \mathrm{Re}[-ik(c_R+ic_I)] = kc_I$ so instability is if $\mathrm{Im}\, c>0$

The wave speed is $-\mathrm{Im}\dfrac{[-ik(c_R+ic_I)]}{k} = c_R = \mathrm{Re}\, c$]

and then

$$(-ikc+ikz)\left[-(k^2+n^2\pi^2)A + \frac{1}{S}A''\right] = 0$$

whence $(z-c)(A''-\mu^2 A)=0 \qquad \mu^2 = (k^2+n^2\pi^2)S \quad (>0)$

& the b.c. is

$$-ikA + ik(z-c)A' = 0$$

~~$A(-ikc+ikz)A'=0,\ z=0,1$~~

whence [redacted]

~~Thus~~

$-A + (z-c)A' = 0$ on $z=0,1$ (we exclude the trivial case $k=0$)

If ~~c~~ $0<c<1$ then ~~therefore~~

[redacted]

~~However~~ there are solutions

$$A''-\mu^2 A = \delta(z-c)$$

which ~~gives~~ gives the Green's function $G(z,c)$ for the o.d.e. & homogeneous b.c's.

Since c is then real, these solutions are not unstable.

[redacted],

The general solution is

$$A = a\cosh\mu z + b\sinh\mu z$$

$$\Rightarrow A' = \mu\left[a\sinh\mu z + b\cosh\mu z\right]$$

on $z=0$, $cA' + A = 0$

$$\Rightarrow \mu c b + a = 0$$

on $z=1$, write $c_h = \cosh\mu$, $s_h = \sinh\mu$, $(1-c)A' = A$

Then $(1-c)\mu\left[as_h + bc_h\right] = ac_h + bs_h$

& since $a = -\mu c b$,

$$\Rightarrow (1-c)\mu\left[-\mu c s_h + c_h\right] = -\mu c c_h + \cancel{b} s_h$$

$$\Rightarrow \mu^2 s_h(c^2 - c) + \mu c_h(1-c) + \mu c_h c - s_h = 0$$

$$\Rightarrow c^2 - c = \frac{s_h - \mu c_h}{\mu^2 s_h}$$

so $c = \frac{1}{2} \pm \left[\frac{1}{4} + \frac{s_h - \mu c_h}{\mu^2 s_h}\right]^{1/2}$.

Comparing this with answer, we write

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 + 1 + \frac{s_h - \mu c_h}{s_h} - 1\right]^{1/2}$$

$$= \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 + 1 - \mu\coth\mu\right]^{1/2}$$

Now note that

$$\frac{1}{2}\left(\coth \frac{\mu}{2} + \tanh \frac{\mu}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\cosh \frac{\mu}{2}}{\sinh \frac{\mu}{2}} + \frac{\sinh \frac{\mu}{2}}{\cosh \frac{\mu}{2}}\right]$$

$$= \frac{\cosh^2 \frac{\mu}{2} + \sinh^2 \frac{\mu}{2}}{2 \sinh \frac{\mu}{2} \cosh \frac{\mu}{2}}$$

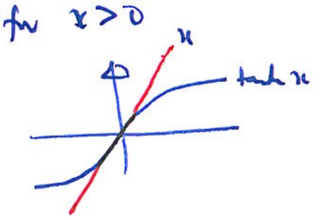
$$= \frac{\cosh \mu}{\sinh \mu} = \coth \mu$$

thus $c = \frac{1}{2} \pm \frac{1}{\mu}\left[\frac{1}{4}\mu^2 + 1 - \frac{\mu}{2}\left(\coth \frac{\mu}{2} + \tanh \frac{\mu}{2}\right)\right]^{\frac{1}{2}}$

$$= \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth \frac{\mu}{2}\right)\left(\frac{\mu}{2} \ \tanh \frac{\mu}{2}\right)\right]^{\frac{1}{2}}$$

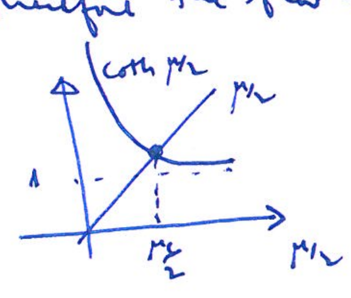
The zonal flow is unstable if $\mathrm{Im}\, c > 0$ & since there are two conjugate roots, the sufficient condition is that $[\ ] < 0$ in above.

Now μ is real and positive w.l.o.g. Also $\tanh x < x$ for $x > 0$



so $\frac{\mu}{2} - \tanh \frac{\mu}{2} > 0$.

Therefore the flow is unstable iff $\frac{\mu}{2} < \coth \frac{\mu}{2}$



viz iff $\mu < \mu_c$, where $\frac{\mu_c}{2} = \coth \frac{\mu_c}{2}$

$(\mu_c \simeq 2 \cdot 4)$

So flow is unstable for

$$\mu = (k^2 + n^2\pi^2)^{\frac{1}{2}} S^{\frac{1}{2}} < \mu_c$$

viz $S < \frac{\mu_c^2}{k^2 + n^2\pi^2}$. Excluding $n = 0$ (trivial) $\Rightarrow S < S_c = \frac{\mu_c^2}{\pi^2}$ for $n = 1$ & $k \to 0$ (longwavelength)

4/

$$(\alpha\rho_s)_t + (\alpha\rho_s v)_z = \Gamma$$

$$[\rho_\ell(1-\alpha)]_t + [\rho_\ell(1-\alpha)u]_z = -\Gamma$$

$$\rho_s(v_t + v v_z) = -p_z - M$$

$$\rho_\ell(u_t + D_\ell u u_z) = -p_z + M$$

If written in the form $A\underline{\Psi}_t + B\underline{\Psi}_z = \underline{c}$

First if $\underline{\Psi}$ was scalar, the characteristics would be $\dot{z} = \frac{B}{A} = A^{-1}B$

& this generalises to systems:

Suppose ~~$A^{-1}Bw = \lambda$~~ $A^{-1}B\underline{w} = \lambda\underline{w}$

So $\det(A^{-1}B - \lambda) = 0$

& suppose $A^{-1}B$ is diagonalisable,

$$A^{-1}B = PDP^{-1}, \quad D = \mathrm{diag}(\lambda_i)$$

Put $\underline{\Psi} = P\underline{u}$

then $A(P\underline{u}_t + P_t\underline{u}) + BP\underline{u}_z = \underline{c}$

$$\Rightarrow \quad \underline{u}_t + P^{-1}A^{-1}BP\underline{u}_z = P^{-1}A^{-1}\underline{c} - P^{-1}A^{-1}AP_t\underline{u}$$

i.e. $\underline{u}_t + D\underline{u}_z = P^{-1}A^{-1}\underline{c} - P^{-1}P_t\underline{u}$

& the characteristics are $\dot{z} = \lambda_i$

on which $\dot{u}_i = [P^{-1}A^{-1}\underline{c} - P^{-1}P_t\underline{u}]_i$

More generally, $\det(\lambda A - B) = 0$.

(i) ρ_g, ρ_l constant, $\dfrac{\rho_g}{\rho_l} = \delta << 1$

Thus

$$\alpha_t + v\alpha_z + \alpha v_z = \frac{\Gamma}{\rho_g}$$

$$-\alpha_t - u\alpha_z + (1-\alpha)u_z = -\frac{\Gamma}{\rho_l}$$

$$\rho_g v_t + \rho_g v v_z + p_z = -M$$

$$\rho_l u_t + D_l \rho_l u u_z + p_z = M$$

& with $\underline{\psi} = \begin{pmatrix} \alpha \\ u \\ v \\ p \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 \\ 0 & \rho_l & 0 & 0 \end{pmatrix}}_{A} \underline{\psi}_t + \underbrace{\begin{pmatrix} v & 0 & \alpha & 0 \\ -u & (1-\alpha) & 0 & 0 \\ 0 & 0 & \rho_g v & 1 \\ 0 & D_l \rho_l u & 0 & 1 \end{pmatrix}}_{B} \underline{\psi}_z = \begin{pmatrix} \Gamma/\rho_g \\ -\Gamma/\rho_l \\ -M \\ M \end{pmatrix}$$

$\det(\lambda A - B)$

$$= \begin{vmatrix} \lambda - v & 0 & -\alpha & 0 \\ -(\lambda - u) & -(1-\alpha) & 0 & 0 \\ 0 & 0 & \rho_g(\lambda - v) & -1 \\ 0 & \rho_l(\lambda - D_l u) & 0 & -1 \end{vmatrix} = 0$$

This gives

$$(\lambda - v)\left[-(1-\alpha)\rho_g(\lambda - v).-1 \right]$$

$$-\alpha\left[-(\lambda - u)\rho_\ell(\lambda - D_\ell u) \right] = 0$$

ie $\rho_g(1-\alpha)(\lambda - v)^2 + \rho_\ell \alpha(\lambda - u)(\lambda - D_\ell u) = 0$

with $\delta = \frac{\rho_g}{\rho_\ell} \ll 1$

$$\delta(1-\alpha)(\lambda - v)^2 + \alpha(\lambda - u)(\lambda - D_\ell u) = 0$$

$$(\lambda - u)(\lambda - D_\ell u) = -\frac{\delta(1-\alpha)}{\alpha}(\lambda - v)^2$$

Approximately $\lambda = u$ & $\lambda = D_\ell u$ are the (real) roots

& corrections eg to u

$$\lambda - u = -\frac{\delta(1-\alpha)}{\alpha}\frac{(\lambda - v)^2}{\lambda - D_\ell u}$$

$$\Rightarrow \lambda \simeq u + \frac{\delta(1-\alpha)(u-v)^2}{\alpha(D_\ell - 1)u} \quad \text{etc.}$$

This only goes wrong if $D_\ell \sim \delta$.

(ii) Now suppose $\frac{d\rho_s}{dp} = \frac{1}{c_g^2}$, $\frac{d\rho_l}{dp} = \frac{1}{c_l^2}$

(c^s are sound speeds)

Then equations are

$$\rho_g \alpha_t + \frac{\alpha}{c_g^2} p_t + \rho_g v \alpha_z + \rho_g \alpha v_z + \frac{\alpha v}{c_g^2} p_z = \Gamma$$

$$-\rho_l \alpha_t + \frac{(1-\alpha)}{c_l^2} p_t \overset{-}{\cancel{+}} \rho_l u \alpha_z + \rho_l (1-\alpha) u_z + \frac{(1-\alpha) u}{c_l^2} p_z = -\Gamma$$

or (define $\underline{\Pi_k = \rho_k c_k^2}$)

$$\alpha_t + \underline{\frac{\alpha}{\Pi_g}} p_t + v \alpha_z + \alpha v_z + \underline{\frac{\alpha v}{\Pi_g}} p_z = \frac{\Gamma}{\rho_s}$$

$$-\alpha_t + \underline{\frac{1-\alpha}{\Pi_l}} p_t - u \alpha_z + (1-\alpha) u_z + \underline{\frac{(1-\alpha) u}{\Pi_l}} p_z = -\frac{\Gamma}{\rho_l}$$

And $\rho_g v_t + \rho_g v v_z + p_z = -M$

$$\rho_l u_t + \rho_l D_l u u_z + p_z = M$$

The red terms are the additions, thus

$$A = \begin{pmatrix} 1 & 0 & 0 & \frac{\alpha}{\Pi_g} \\ -1 & 0 & 0 & \frac{1-\alpha}{\Pi_l} \\ 0 & 0 & \rho_s & 0 \\ 0 & \rho_l & 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} v & 0 & \alpha & \frac{\alpha v}{\Pi_g} \\ -u & (1-\alpha) & 0 & \frac{(1-\alpha) u}{\Pi_l} \\ 0 & 0 & \rho_g v & 1 \\ 0 & D_l \rho_l u & 0 & 1 \end{pmatrix}$$

and thus

$$\det(\lambda A - B)$$

$$= \begin{vmatrix} \lambda - v & 0 & -\alpha & \frac{\alpha}{\Pi_g}(\lambda - v) \\ -(\lambda - u) & -(1-\alpha) & 0 & \frac{(1-\alpha)}{\Pi_\ell}(\lambda - u) \\ 0 & 0 & \rho_g(\lambda - v) & -1 \\ 0 & \rho_\ell(\lambda - D_\ell u) & 0 & -1 \end{vmatrix} = 0$$

If $(\lambda - v)\left[(1-\alpha)\rho_g(\lambda - v) - \overset{\downarrow}{\frac{(1-\alpha)}{\Pi_\ell}}(\lambda - u)\rho_g\rho_\ell(\lambda - v)(\lambda - D_\ell u)\right]$

$$- \alpha\left[-(\lambda - u)\rho_\ell(\lambda - D_\ell u)\right]$$

$$- \frac{\alpha}{\Pi_g}\overset{\downarrow}{(\lambda - v)}\left[-(\lambda - u) . -\rho_g\rho_\ell(\lambda - v)(\lambda - D_\ell u)\right] = 0$$

i.e $\rho_g(1-\alpha)(\lambda - v)^2 + \rho_\ell\alpha(\lambda - u)(\lambda - D_\ell u)$

$$- \rho_g\rho_\ell\left[\frac{1-\alpha}{\Pi_\ell} + \frac{\alpha}{\Pi_g}\right](\lambda - v)^2(\lambda - u)(\lambda - D_\ell u) = 0$$

Previously, we had $\lambda \sim u$

Let's suppose $v \sim u$ and $c_g \sim c_\ell$

If $\lambda \sim u$ the the new term is $\sim \left[\frac{\rho_g}{c_\ell^2} , \frac{\rho_\ell}{c_g^2} \right] u^4 \ll \rho_\ell u^2$

So two approximate roots are $\lambda = u, D_\ell u$ as before.

As the new term is singular, the other two roots are large

Approximately $$\rho_\ell \alpha (\lambda - u)(\lambda - D_\ell u) - \frac{\alpha \rho_\ell}{c_g^2} (\lambda - v)^2 (\lambda - u)(\lambda - D_\ell u) \simeq 0$$

So the other two roots are

$\lambda = v \pm c_g$ gas sound waves

5)

Energy equation is

$$\Gamma L + \alpha\rho_g c_{pg}(T_t + vT_z) + (1-\alpha)\rho_l c_{pl}(T_t + uT_z)$$

$$-\{(\alpha p_g)_t + (\alpha p_g v)_z\} - [\{(1-\alpha)p_l\}_t + \{(1-\alpha)p_l u\}_z] = Q$$

where $\Gamma = (\alpha\rho_g)_t + (\alpha\rho_g v)_z = -[\{(1-\alpha)\rho_l\}_t + \{(1-\alpha)\rho_l u\}_z]$

Energy is thus

$$\Gamma L + \alpha\rho_g(h_{gt} + vh_{gz}) + (1-\alpha)\rho_l(h_{lt} + uh_{lz})$$

$$-\{(\alpha p_g)_t + (\alpha p_g v)_z\} - [\{(1-\alpha)p_l\}_t + \{(1-\alpha)p_l u\}_z] = Q$$

~~Now note that $(\alpha\rho_g)_t + (\alpha\rho_g v)_z$~~

viz $\Gamma(h_g - h_l) + \dots$

viz $h_g[(\alpha\rho_g)_t + (\alpha\rho_g v)_z] + \alpha\rho_g(h_{gt} + vh_{gz})$

$+ h_l[\{(1-\alpha)\rho_l\}_t + \{(1-\alpha)\rho_l u\}_z] + (1-\alpha)\rho_l(h_{lt} + uh_{lz})$

$- \dots = Q$

viz $(\alpha\rho_g h_g)_t + (\alpha\rho_g v h_g)_z - (\alpha p_g)_t - (\alpha p_g v)_z$

$+ \{(1-\alpha)\rho_l h_l\}_t + \{(1-\alpha)\rho_l h_l u\}_z - \{(1-\alpha)p_l\}_t - \{(1-\alpha)p_l u\}_z = Q$

viz $\{\alpha(\rho_g h_g - p_g)\}_t + \{\alpha v(\rho_g h_g - p_g)\}_z$

$+ \{(1-\alpha)(\rho_l h_l - p_l)\}_t + \{(1-\alpha)(\rho_l h_l - p_l)u\}_z = Q$

viz $(\alpha\rho_g e_g)_t + (\alpha\rho_g e_g v]_z$

$+ \{(1-\alpha)\rho_\ell e_\ell\}_t + \{(1-\alpha)\rho_\ell e_\ell u\}_z = Q$ □

$\rho = \rho_\ell(1-\alpha) + \rho_g\alpha$

$p = (1-\alpha)p_\ell + \alpha p_g$

$\rho e = \alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell$

$h = e + \frac{p}{\rho}$

$$\Rightarrow \rho h = \rho e + p = \alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell + \alpha p_g + (1-\alpha)p_\ell$$
$$= \alpha\rho_g h_g + (1-\alpha)\rho_\ell h_\ell$$ □

Now suppose $u = v$ (homogeneous flow)

The energy equation is then

$$\{\alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell\}_t + [\{\alpha\rho_g e_g + (1-\alpha)\rho_\ell e_\ell\}u]_z = Q$$

i.e $(\rho e)_t + (\rho e u)_z = Q$

& $\rho_t + (\rho u)_z = 0$ so $\underline{\rho\frac{de}{dt} = Q}$, $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial z}$

We have $h = e + \frac{p}{\rho}$. When $\alpha = 0$, $\rho = \rho_\ell$, $e = e_\ell$, $p = p_\ell$

$\Rightarrow h = h_\ell$

Sim. at $\alpha = 1$ $h = h_g$

So as $\alpha : 0 \rightarrow 1$ $\quad \Delta h \sim h_g - h_l = L$

and $\Delta h = \Delta e + \Delta\left(\frac{p}{\rho}\right)$

$$\Delta\left(\frac{p}{\rho}\right) < \frac{\Delta p}{\rho_g}$$

So if $\Delta p \ll \rho_g L$ then $\Delta\left(\frac{p}{\rho}\right) \ll \Delta h$

$$\Rightarrow \underline{h \simeq e}$$

In this case $\rho \frac{dh}{dt} \simeq Q$

Now $\quad \rho = \rho_l - \Delta\rho\, \alpha \qquad \Delta\rho = \rho_l - \rho_g$

$$\Rightarrow \alpha = \frac{\rho_l - \rho}{\Delta\rho}$$

& $\rho h = \alpha \rho_g h_g + (1-\alpha)\rho_l h_l$

$$= \rho_l h_l + (\rho_g h_g - \rho_l h_l)\frac{(\rho_l - \rho)}{\Delta\rho}$$

$$= \frac{\rho_l h_l(\rho_l - \rho_g) + (\rho_g h_g - \rho_l h_l)\rho_l - (\rho_g h_g - \rho_l h_l)\rho}{\Delta\rho}$$

thus $h = \frac{\rho_l \rho_g L}{\Delta\rho}\frac{1}{\rho} - \frac{(\rho_g h_g - \rho_l h_l)}{\Delta\rho}$

then $\rho \frac{dh}{dt} = \frac{\rho_l \rho_g L}{\Delta\rho} \cdot \rho \cdot -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{\rho_l \rho_g L}{\Delta\rho} u_z = Q$ $\qquad$ (note $\frac{d\rho}{dt} = -\rho u_z$)

$$\Rightarrow \underline{u_z = \frac{(\rho_l - \rho_g)Q}{\rho_l \rho_g L}}$$

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$$\rho_t + u\rho_z = -u_z\rho$$

$$\rho(u_t + uu_z) = -p_z - \rho g - \frac{4f\rho_l u^2}{d}$$

$$\rho(h_t + uh_z) = Q$$

$$h \simeq h^* + \frac{\rho_g L}{\rho} \quad (\text{if } \alpha > 0)$$

$z = 0 \quad h = h_0, \; u = U(t)$

$z = r \quad h = h_{sat}$

For $z < r$, $\rho = \rho_l$, $h_t + uh_z = \dfrac{Q}{\rho_l}$, $u = U(t)$ (as $u_z = 0$)

characteristics $\dot{z} = U$, $\dot{h} = \dfrac{Q}{\rho_l}$, $z = 0, t = \eta, h = h_0$

$$\Rightarrow \quad h = \frac{Q}{\rho_l}(t - \eta) + h_0, \quad z = \int_\eta^t U(s)\,ds$$

on $z = r(t)$, $h = h_{sat}$ $\Rightarrow$ $\eta = t - \dfrac{\rho_l(h_{sat} - h_0)}{Q} = t - \tau$, $\underline{\tau = \dfrac{\rho_l(h_{sat} - h_0)}{Q}}$

thus $\underline{r = \displaystyle\int_{t-\tau}^t U(s)\,ds}$

In the single-phase region the pressure drop is

$$\Delta p_{sp} = \int_0^r -p_z\,dz = \int_0^r \left[\rho_l \dot{U} + \rho_l g + \frac{4f\rho_l U^2}{d}\right] dz$$

$$= \left[\rho_l \dot{U} + \rho_l g + \frac{4f\rho_l U^2}{d}\right] r$$

Non-dimensionalise via

$$\rho \sim \rho_l \,, \quad z, r \sim l \,, \quad t \sim \tau, \quad u, U \sim u_0 = \frac{l}{\tau}$$

(non-d)

$$\Rightarrow \Delta p_{sp} = \left[\rho_l u_0^2 \dot{U} + \rho_l g l + \frac{4 f l \rho_l u_0^2}{d} U^2\right] r$$

$$= \underline{\left[\Delta p_i \dot{U} + \Delta p_g + \Delta p_f U^2\right] r} \quad , \text{also } \underline{r = \int_{t-1}^{t} U(s)\, ds}$$

$\Delta p_i = \rho_l u_0^2$

$\Delta p_g = \rho_l g l$

$\Delta p_f = \dfrac{4 f l \rho_l u_0^2}{d}$

For $z > r$ (all non-d)

$$\rho_t + u \rho_z = -u_z \rho$$

& also (dimensional): $\rho \cdot \rho_g L \cdot -\dfrac{1}{\rho^2}\dfrac{d\rho}{dt} = Q$

or $\rho_g L u_z = Q$ $\quad \left(\dfrac{d\rho}{dt} = -u_z \rho \right)$

& non-d $\quad u_z = \dfrac{Q}{\rho_g L}\dfrac{l}{u_0} = \dfrac{Q\tau}{\rho_g L} = \dfrac{\rho_l \Delta h}{\rho_g L}$, $\quad \Delta h = h_{sat} - h_0$

Define $\underline{\varepsilon = \dfrac{\rho_g L}{\rho_l \Delta h}}$ then $u_z = \dfrac{1}{\varepsilon}$, $u = U$ at $z = r$

$$\Rightarrow \underline{u = U + \frac{z - r}{\varepsilon}}$$

The characteristics for ρ are thus (non-d)

$$\dot{z} = u = U + \frac{z-r}{\varepsilon}$$

$$\dot{\rho} = -\frac{1}{\varepsilon}\rho$$

& $t=\eta$, $z=r(\eta)$, $\rho=1$

giving $\rho = \exp\left[-(t-\eta)/\varepsilon\right] \Rightarrow \eta = t - \varepsilon \ln\frac{1}{\rho}$

$$\dot{z} = U + \frac{z-r}{\varepsilon}$$

$$\dot{z} - \frac{z}{\varepsilon} = U - \frac{r}{\varepsilon}$$

$$\left(z e^{-t/\varepsilon}\right)^{\dot{}} = \left(U - \frac{r}{\varepsilon}\right) e^{-t/\varepsilon}$$

$$z e^{-t/\varepsilon} = r(\eta) e^{-\eta/\varepsilon} + \int_\eta^t \left\{U(s) - \frac{r(s)}{\varepsilon}\right\} e^{-s/\varepsilon} ds$$

i.e. $$z = r(\eta) e^{\frac{t-\eta}{\varepsilon}} + \int_\eta^t \left\{U(s) - \frac{r(s)}{\varepsilon}\right\} e^{\frac{t-s}{\varepsilon}} ds$$

writing $s = t - \varepsilon\xi$, $\eta = t - \varepsilon \ln\frac{1}{\rho}$, note when $s=\eta$, $\xi = \ln\frac{1}{\rho}$

$$z = r\left(t - \varepsilon \ln\frac{1}{\rho}\right)\frac{1}{\rho} + \int_0^{\ln\frac{1}{\rho}} \left\{\varepsilon U(t-\varepsilon\xi) - r(t-\varepsilon\xi)\right\} e^{\xi} d\xi$$

Integrating by parts, $\int_0^{\ln\frac{1}{\rho}} r(t-\varepsilon\xi) e^{\xi} d\xi$

$$= \left[e^{\xi} r(t-\varepsilon\xi)\right]_0^{\ln\frac{1}{\rho}} + \varepsilon\int_0^{\ln\frac{1}{\rho}} e^{\xi} r'(t-\varepsilon\xi)\, d\xi$$

$$= r\left(t - \varepsilon \ln\frac{1}{\rho}\right)\frac{1}{\rho} - r(t) + \varepsilon\int_0^{\ln\frac{1}{\rho}} e^{\xi}\left[U(t-\varepsilon\xi) - U_1(t-\varepsilon\xi)\right] d\xi$$

$\{U_1(t) = U(t-1)\}$

Thus

$$z = \dot{r}\left(t-\varepsilon \ln\tfrac{1}{\rho}\right)\cdot\frac{1}{\rho} + \int_0^{\ln\frac{1}{\rho}} \varepsilon \dot{U}(t-\varepsilon\xi)e^{\xi}d\xi$$

$$- \left[\dot{r}\left(t-\varepsilon \ln\tfrac{1}{\rho}\right)\cdot\frac{1}{\rho} - r(t) + \varepsilon\int_0^{\ln\frac{1}{\rho}} e^{\xi}\left[\dot{U}(t-\varepsilon\xi) - U_1(t-\varepsilon\xi)\right]d\xi\right]$$

$$= r(t) + \varepsilon\int_0^{\ln\frac{1}{\rho}} U_1(t-\varepsilon\xi)e^{\xi}\,d\xi$$

In dimensional terms, the two-phase pressure drop is

$$\Delta p_{tp} = \int_r^z \left[(\rho u)_t + (\rho u^2)_z + \rho g + \frac{4f\rho}{d}u^2\right]dz$$

& the RHS in dimensionless terms $(\rho_0 u)$ is

$$\Delta p_i \int_r^1 \underbrace{[(\rho u)_t + (\rho u^2)_z}_{\Phi_i}]dz + \Delta p_g \int_r^1 \underbrace{\rho}_{\Phi_g}\,dz + \Delta p_f \int_r^1 \underbrace{\rho u^2}_{\Phi_f}\,dz$$

Steady state

$U = V$, $r = V$. We need $V < 1$ so there is a two-phase region!

We have in the two-phase region $u = V + \dfrac{z-V}{\varepsilon}$, & $\rho u = V$

Then $\Delta p = \Delta p_{rp} + \Delta p_{tp}$

$$= (\Delta p_g + \Delta p_f V^2)V$$

$$+ \Delta p_i [\cancel{\rho} u]_r^1 + \Delta p_g \int_r^1 \frac{V\,dz}{u} + \Delta p_f \int_r^1 u\,dz$$

viz $\Delta p = (\Delta p_s + \Delta p_f V^2) V$

$$+ \Delta p_i \cdot \frac{(1-V)}{\varepsilon} + \Delta p_g V \varepsilon \ln\left[\frac{V + \frac{(1-V)}{\varepsilon}}{V}\right] + \Delta p_f \left[Vz + \frac{(z-V)^2}{2\varepsilon}\right]_V^1$$

viz

$$\Delta p = \Delta p_i \cdot \frac{(1-V)}{\varepsilon} + \Delta p_g V\left[1 + \varepsilon \ln\left\{1 + \frac{(1-V)}{\varepsilon V}\right\}\right]$$

$$+ \Delta p_f\left[V^3 + V(1-V) + \frac{(1-V)^2}{2\varepsilon}\right]$$

If V is sufficiently close to 1

$\Delta p \sim \Delta p_g V + \Delta p_f V^3$ $\qquad \uparrow$ with V

small ε

$\Delta p \simeq \frac{1}{\varepsilon}\left[\Delta p_i (1-V) + \frac{1}{2}\Delta p_f (1-V)^2\right]$ $\qquad \downarrow$ with V (<1)

[This is the Ledinegg instability, where $\frac{d\Delta p}{dV} < 0$.]

Next, $\Delta p_i = \Delta p_g = 0$, thus

$$\frac{\Delta p}{\Delta p_f} = \Pi, \text{ say}, = U^2 r + \int_r^1 \rho u^2 \, dz$$

single phase

two phase

Linear stability

with

$$U = V + v\,, \quad r = r_0 + r_1\,, \quad u = u_0 + u_1\,, \quad \rho = \rho_0 + \rho_1$$

we have $\quad r_0 = V\,, \quad u_0 = V + \frac{z-V}{\varepsilon}\,, \quad \rho_0 = \frac{V}{u_0} = \frac{V}{V + \frac{z-V}{\varepsilon}}$

$$r = \int_{t-1}^{t} U(s)\,ds$$

so $\quad r_1 = \int_{t-1}^{t} v\,ds$

$$u = U + \frac{z-r}{\varepsilon} \Rightarrow u_1 = v - \frac{r_1}{\varepsilon}$$

Note also

$$z = r(t) + \varepsilon \int_0^{\ln\frac{1}{\rho}} U_1(t-\varepsilon\xi)e^{\xi}\,d\xi$$

$$= r_0 + r_1 + \varepsilon \int_0^{-\ln\rho_0 - \frac{\rho_1}{\rho_0}\cdots} \{V + v_1(t-\varepsilon\xi)\}e^{\xi}\,d\xi$$

$$\left[\begin{aligned} \ln\frac{1}{\rho} &= -\ln\rho \\ &= -\ln(\rho_0+\rho_1) \\ &= -\ln\rho_0 - \frac{\rho_1}{\rho_0}\cdots \end{aligned}\right.$$

linearising

$$0 = r_1 + \varepsilon \int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)e^{\xi}\,d\xi$$

$$-\frac{\varepsilon\rho_1}{\rho_0}\frac{V}{\rho_0}$$

$$\Rightarrow \rho_1 \approx \frac{\rho_0^2}{V}\left[\frac{r_1}{\varepsilon} + \int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)e^{\xi}\,d\xi\right]$$

Linearising the condition $\overset{= U^2 r + \int_r^1 \rho u^2 dz}{\frac{\Delta p}{\Delta p_f}} =$ constant, we have

$$0 = V^2 r_1 + 2V^2 v - r_1 \rho_0 u_0^2\Big|_{r_0} + \int_{r_0}^{1} \rho_1 u_0^2\,dz + \int_{r_0}^{1} 2\rho_0 u_0 u_1\,dz$$

$$= \overbrace{V^2 r_1 + 2V^2 v}^{\text{single phase}} - V^2 r_1 + \int_{r_0}^{1}(\rho_1 u_0^2 + 2Vu_1)\,dz$$

ie $\quad \underset{\text{s.p.}}{2V^2 v} + \int_{r_0}^{1}(\rho_1 u_0^2 + 2Vu_1)\,dz = 0$

Since $\rho_0 u_0 = V$,

$$\rho_1 u_0^2 = V\left[\frac{r_1}{\varepsilon} + \int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)\, e^{\xi}\, d\xi\right]$$

So

$$0 = 2V^2 v \;\; [+V^2 r_1 \quad -V^2 r_1] + \int_{r_0}^{1} V\left[2u_1 + \frac{r_1}{\varepsilon} + \int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)\, e^{\xi}\, d\xi\right] dz$$

s.p.

Now we put $v = e^{\sigma t}$;

then $r_1 = \frac{1}{\sigma}(1-e^{-\sigma})\, e^{\sigma t}$

$u_1 = \left[1 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})\right] e^{\sigma t}$

We have

$$\int_0^{\ln\frac{1}{\rho_0}} v_1(t-\varepsilon\xi)\, e^{\xi}\, d\xi = \int_0^{\ln\frac{1}{\rho_0}} e^{\sigma(t-1-\varepsilon\xi)}\, e^{\xi}\, d\xi$$

$$= e^{\sigma t} \int_0^{\ln\frac{1}{\rho_0}} e^{(1-\varepsilon\sigma)\xi}\, d\xi \times e^{-\sigma}$$

$$= e^{\sigma t}\left[\frac{e^{-\sigma}}{1-\varepsilon\sigma}\left\{\frac{1}{\rho_0^{1-\varepsilon\sigma}} - 1\right\}\right]$$

$$= \frac{e^{\sigma t} e^{-\sigma}}{1-\varepsilon\sigma}\left[\left(1 + \frac{z-V}{\varepsilon V}\right)^{1-\varepsilon\sigma} - 1\right],$$

so

$$2V^2 \;\; [+V^2 r_1 - V^2 r_1] \; + \; V(1-V)\left[\overbrace{2\left\{1 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})\right\} + \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})}^{2 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma})}\right]$$

$$+ \frac{Ve^{-\sigma}}{1-\varepsilon\sigma}\int_V^1 \left\{\left(1 + \frac{z-V}{\varepsilon V}\right)^{1-\varepsilon\sigma} - 1\right\} dz = 0$$

s.p.

viz

$$\underbrace{2V^2 \quad \left[+ V^2 \frac{1}{\sigma}(1-e^{-\sigma})\right.}_{\text{single phase } 0<z<r} \quad \left. - V^2 \frac{1}{\sigma}(1-e^{-\sigma}) \right]$$

$$+ \quad V(1-V)\left[2 - \frac{1}{\varepsilon\sigma}(1-e^{-\sigma}) \right]$$

$$+ \frac{Ve^{-\sigma}}{(1-\varepsilon\sigma)}\left[\frac{\varepsilon V}{2-\varepsilon\sigma}\left\{ \left(1+\frac{1-V}{\varepsilon V}\right)^{2-\varepsilon\sigma} - 1 \right\} - (1-V) \right] = 0$$

(9) only include the single-phase pressure drop :

$$\Rightarrow \quad 2V^2 \quad + V^2 \frac{1}{\sigma}(1-e^{-\sigma}) = 0$$

Then $\frac{1}{\sigma}(1-e^{-\sigma}) + 2 = 0$

$$\Rightarrow \quad \sigma = -\tfrac{1}{2}(1-e^{-\sigma})$$

Suppose $\mathrm{Re}\,\sigma > 0$: then $|e^{-\sigma}| < 1$, so $\mathrm{Re}\,(1-e^{-\sigma}) > 0$

so $\mathrm{Re}\,\sigma = -\tfrac{1}{2}\mathrm{Re}\,(1-e^{-\sigma}) < 0 \; \times \quad \Rightarrow \mathrm{Re}\,\sigma < 0$

(note $\sigma = 0$ is always a root : since $\frac{\Delta p}{\Delta p_f} = U^2(t)\int_{t-1}^{t} U(s)\,ds$

this is associated with time translation invariance .)

(b) only two-phase pressure drop :

$$\Rightarrow \quad -V^2\frac{1}{\sigma}(1-e^{-\sigma}) + 2V(1-V) - \overset{\sim\frac{1}{\varepsilon}}{V(1-V)\frac{1}{\varepsilon\sigma}(1-e^{-\sigma})}$$

$$+ \frac{Ve^{-\sigma}}{1-\varepsilon\sigma}\left[\underbrace{\frac{\varepsilon V}{2-\varepsilon\sigma}\left\{\left(1+\frac{1-V}{\varepsilon V}\right)^{2-\varepsilon\sigma}\right.}_{\sim\frac{1}{\varepsilon}}\left. -1\right\} - (1-V)\right] = 0$$

$\varepsilon \ll 1$,

$$\text{approx} \quad -\frac{V(1-V)}{\varepsilon}\frac{1}{\sigma}(1-e^{-\sigma}) + Ve^{-\sigma}\frac{\varepsilon V}{2}\left(\frac{1-V}{\varepsilon V}\right)^2 = 0$$

$$\Rightarrow \quad -\frac{V(1-V)}{V^2}\frac{2V^2}{(1-V)^2}(e^{\sigma}-1) + \sigma = 0$$

$$\Rightarrow \quad \sigma = \gamma(e^{\sigma}-1), \quad \gamma = \frac{2V}{1-V} > 0$$

As $\sigma \to \infty$ (LHS) we must have $e^{\sigma} \to \infty \Rightarrow \mathrm{Re}\,\sigma > 0$ & $\to \infty$

⇒ ill-posedness

(c) Use both pressure drops & the small ε approx

$$\Rightarrow 2V^2 - \frac{V(1-V)}{\varepsilon}\frac{1}{\sigma}(1-e^{-\sigma}) + \underset{\uparrow \atop e^{-\sigma}\frac{(1-V)^2}{2\varepsilon}}{\frac{\varepsilon V^2 e^{-\sigma}}{2}\frac{(1-V)^2}{\varepsilon^2 V^2}} \approx 0$$

$$\Rightarrow \quad \frac{4\varepsilon V^2}{(1-V)^2} - \gamma\frac{1}{\sigma}(1-e^{-\sigma}) + e^{-\sigma} = 0$$

Define $\delta = \frac{4\varepsilon V^2}{(1-V)^2}$, $\Rightarrow \delta\sigma - \gamma + \gamma e^{-\sigma} + \sigma e^{-\sigma} = 0$

$$\Rightarrow \quad \sigma = \frac{\gamma(1-e^{-\sigma})}{\delta + e^{-\sigma}}$$

Now as $\sigma \to \infty$ (LHS) we must have

$$e^{-\sigma} \to -\delta$$

so $e^{\sigma} \to -\frac{1}{\delta}$

$$\sigma \to \ln\frac{1}{\delta} + (2n+1)i\pi$$

is ill-posed for $\delta < 1$ (as then $\operatorname{Re}\sigma > 0$ as $n \to \infty$)

(d) If we also include the inertial term in the single-phase region

$\Delta p_i \dot{U}$, then (c) is modified to

$$\frac{\Delta p_i}{\Delta p_f}\sigma + 2V^2 - \frac{V(1-V)}{\varepsilon}\frac{1}{\sigma}(1-e^{-\sigma}) + \frac{(1-V)^2}{2\varepsilon}e^{-\sigma} = 0$$

$$\Rightarrow \nu\sigma + \delta - \gamma\frac{1}{\sigma}(1-e^{-\sigma}) + e^{-\sigma} = 0, \quad \nu = \frac{2\varepsilon\Delta p_i}{(1-V)^2\Delta p_f}$$

$$\Rightarrow \nu\sigma^2 + \sigma(\delta + e^{-\sigma}) - \gamma(1-e^{-\sigma}) = 0 \qquad \delta = \frac{4\varepsilon V^2}{(1-V)^2}$$

Now as $\sigma \to \infty$ we must have

$$\sigma(\delta + e^{-\sigma}) \gg \gamma(1-e^{-\sigma})$$

& so we must have

$$\nu\sigma^2 \simeq -\sigma(\delta + e^{-\sigma})$$

i.e. $\nu\sigma \simeq -\delta - e^{-\sigma}$

LHS $\to \infty \Rightarrow e^{-\sigma} \to \infty \Rightarrow \operatorname{Re}\sigma \to -\infty \Rightarrow$ well-posed.

Instability Clearly as $\nu, \delta \to 0$ there is instability because of (b)

Therefore the steady state is unstable for sufficiently small ν & δ, i.e. small enough ε.