Perturbation Methods: Problem Sheet 4

Q1(a) $\ddot{x} + \varepsilon\dot{x} + x = 0$

$x = x(t,T)$, $t = \frac{T}{\varepsilon}$ $\Rightarrow$ $\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}$

$\Rightarrow x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x_t + \varepsilon x_T) + x = 0$

$x \sim x_0(t,T) + \varepsilon x_1(t,T) + \ldots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $x_{0tt} + x_0 = 0 \Rightarrow \underline{x_0 = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(T)e^{-it}\right)}$

$$\begin{aligned} O(\varepsilon^1): \quad x_{1tt} + x_1 &= -2x_{0tT} - x_{0t} \\ &= -\left(iA_T e^{it} - i\bar{A}e^{-it}\right) - \tfrac{1}{2}\left(iAe^{it} - i\bar{A}e^{-it}\right) \\ &= -i\left(A_T + \tfrac{1}{2}A\right)e^{it} + \text{c.c.} \end{aligned}$$

Can suppress secular terms $e^{\pm it}$ only if $\underline{A_T + \frac{1}{2}A = 0}$

$A = Re^{i\Theta} \Rightarrow R_T + iR\Theta_T + \frac{1}{2}R = 0$

$\Rightarrow \Theta_T = 0, \quad R_T = -\frac{1}{2}R$

$\Rightarrow \Theta = \Theta(0), \quad R = R(0)e^{-T/2} \qquad (\Theta(0), R(0) \in \mathbb{R})$

$\Rightarrow \underline{x_0 = R\cos(t + \Theta) = R(0)e^{-T/2}\cos(t + \Theta(0))}$

Exact solution is

$$x = r_0 e^{-\varepsilon t/2}\cos\left[\left(1 - \frac{\varepsilon^2}{4}\right)^{1/2} t + \theta_0\right] \qquad (r_0, \theta_0 \in \mathbb{R})$$

$$\sim r_0 e^{-T/2}\cos\left[t + \theta_0 - \frac{\varepsilon^2 t}{8}\right] \qquad \text{as } \varepsilon \to 0$$

$\Rightarrow x - x_0 = O(\varepsilon)$ for $t = O(1/\varepsilon)$ □

(b) $\ddot{x} + x = \varepsilon x^3 \Rightarrow$ same as (a) until

$O(\varepsilon^1)$: $x_{1tt} + x_1 = -2x_{0tT} - x_0^3$

$$= -(iA_T e^{it} - i\bar{A}_T e^{-it}) - \tfrac{1}{8}(Ae^{it} + \bar{A}e^{-it})^3$$

$$= \left[-iA_T + \tfrac{3}{8}A^2\bar{A}\right]e^{it} + \text{c.c.} + \text{non-secular.}$$

Can suppress secular terms $e^{\pm it}$ only if $iA_T = \frac{3}{8}A^2\bar{A}$

$A = Re^{i\Theta} \Rightarrow i(R_T + iR\Theta_T) = \frac{3}{8}R^3$

$\Rightarrow R_T = 0, \quad R\Theta_T = -\frac{3}{8}R^3$

$\Rightarrow R = R(0), \quad \Theta = \Theta(0) - \frac{3}{8}R(0)^2 T$

$\Rightarrow A(T) = R(0)\, e^{i\Theta(0) - \frac{3i}{8}R(0)^2 T}$

$\Rightarrow A(T) = A(0)\, e^{-\frac{3i}{8}|A(0)|^2 T}$

Q2 $\frac{d}{dx}\left(D(x,\frac{x}{\varepsilon})\frac{du}{dx}\right) = f(x,\frac{x}{\varepsilon})$

$u = u(x, X)$, $x = \varepsilon X \Rightarrow \frac{d}{dx} = \frac{1}{\varepsilon}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$

$$\Rightarrow \left(\frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial x}\right)\left(D(x,X)\left(\frac{\partial u}{\partial X} + \varepsilon\frac{\partial u}{\partial x}\right)\right) = \varepsilon^2 f(x,X)$$

$u \sim u_0(x,X) + \varepsilon u_1(x,X) + \varepsilon^2 u_2(x,X) + \ldots$ as $\varepsilon \to 0$ gives the following problems at $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^2)$.

$O(1)$: $\quad (D u_{0X})_X = 0 \qquad$ (+)

$O(\varepsilon)$: $\quad (D(u_{1X} + u_{0x}))_X + (D u_{0X})_x = 0 \qquad$ (++)

$O(\varepsilon^2)$: $\quad (D(u_{2X} + u_{1x}))_X + (D(u_{1X} + u_{0x}))_x = f \qquad$ (+++)

(+) $\Rightarrow \quad D u_{0X} = a_1(x) \qquad$ (a_1 arb.)

$\Rightarrow \quad u_0 = a_0(x) + a_1(x)\int_0^X \frac{ds}{D(x,s)} \qquad$ (a_0 arb.)

u_0 periodic in X with period 1

$\Rightarrow \quad a_0(x) = u_0(x,0) = u_0(x,1) = a_0(x) + a_1(x)\int_0^1 \frac{ds}{D(x,s)}$

$\Rightarrow \quad a_1 = 0 \quad \because \int_0^1 \frac{ds}{D(x,s)} > 0$

$\Rightarrow \quad u_0 = a_0(x) = u_0(x)$, say.

(++) $\Rightarrow \quad D(u_{1X} + u_{0x}) = b_1(x) \qquad$ (b_1 arb.)

$\Rightarrow \quad u_1 = b_0(x) - u_{0x}X + b_1(x)\int_0^X \frac{ds}{D(x,s)} \qquad$ (b_0 arb.)

u_1 periodic in X with period 1

$\Rightarrow \quad b_0(x) = u_1(x,0) = u_1(x,1) = b_0(x) - u_{0x} + b_1(x)\int_0^1 \frac{ds}{D(x,s)}$

$$\Rightarrow \quad b_1(x) = \hat{D}(x)\, u_{0x} \quad \text{where } \hat{D}(x) := \left({}_0\!\int^1 \frac{ds}{D(x,s)} \right)^{-1},$$

i.e. $\hat{D}$ is the harmonic average of D over one period.

$$(\#) \quad \Rightarrow \quad \left(D(u_{2X} + u_{1x}) \right)_X = f - b_{1x}$$

$$\Rightarrow \quad D(u_{2X} + u_{1x}) = c_1(x) + {}_0\!\int^X f(x,s)\,ds - b_{1x} X$$

$(c_1 \text{ arb.})$

u_1, u_2 periodic in X with period 1

$$\Rightarrow \quad u_{2X},\ u_{1x}(x,X) = \lim_{h\to 0} \frac{u_1(x+h,X) - u_1(x,X)}{h} \quad \text{periodic in } X \text{ with period } 1$$

$$\Rightarrow \quad c_1(x) = D(u_{2X} + u_{1x})\big|_{X=0}$$

$$= D(u_{2X} + u_{1x})\big|_{X=1}$$

$$= c_1(x) + {}_0\!\int^1 f(x,s)\,ds - b_{1x}$$

$$\Rightarrow \quad b_{1x} = {}_0\!\int^1 f(x,s)\,ds$$

Hence, homogenized equation for $u_0(x)$ is

$$\frac{d}{dx}\left(\hat{D}(x) \frac{du_0}{dx} \right) = \hat{f}(x),$$

where

$$\hat{D}(x) = \left({}_0\!\int^1 \frac{ds}{D(x,s)} \right)^{-1}$$

$$\hat{f}(x) = {}_0\!\int^1 f(x,s)\,ds$$

Note different averages for the diffusivity $\hat{D}(x)$ and for the capacity $\hat{f}(x)$.

Q3 Let $y = A(x) e^{iu(x)/\varepsilon}$

$$\Rightarrow \quad y' = \left(\frac{iAu'}{\varepsilon} + A' \right) e^{iu/\varepsilon}$$

$$y'' = \left(-\frac{A(u')^2}{\varepsilon^2} + \frac{2iA'u'}{\varepsilon} + \frac{iAu''}{\varepsilon} + A'' \right) e^{iu/\varepsilon}$$

(a) $\varepsilon^2 y'' + xy = 0$ for $x > 0$

$$\Rightarrow \quad -A(u')^2 + 2i\varepsilon A'u' + i\varepsilon A u'' + \varepsilon^2 A'' + xA = 0$$

$A \sim A_0(x) + \varepsilon A_1(x) + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $-A_0(u')^2 + xA_0 = 0 \underset{(A_0 \neq 0)}{\Rightarrow} u' = \pm x^{1/2}$, $u = \pm \frac{2}{3} x^{3/2}$ (wlog)

$O(\varepsilon^1)$: $-A_1(u')^2 + 2iA_0'u' + iA_0u'' + xA_1 = 0$

$$\Rightarrow \quad 2A_0' x^{1/2} + A_0 \tfrac{1}{2} x^{-1/2} = 0$$

$$\Rightarrow \quad \frac{A_0'}{A_0} = -\frac{1}{4x}$$

$$\Rightarrow \quad \ln|A_0| = c_1 - \frac{1}{4}\ln x \quad (c_1 \in \mathbb{R})$$

$$\Rightarrow \quad A_0 = \frac{c_2}{x^{1/4}} \quad (|c_2| = e^{c_1})$$

Hence, $y_1 \sim \frac{c_2^+}{x^{1/4}} e^{\frac{2ix^{3/2}}{3\varepsilon}}$, $y_2 \sim \frac{c_2^-}{x^{1/4}} e^{-\frac{2ix^{3/2}}{3\varepsilon}}$ as $\varepsilon \to 0^+$ $(c_2^\pm \in \mathbb{R})$

(b) $\varepsilon^2 y'' - xy = 0$ for $x > 0$.

Obtain similarly $u = \pm \frac{2ix^{3/2}}{3}$, $A_0 = \frac{c_2^\pm}{x^{1/4}}$ $(c_2^\pm \in \mathbb{R})$

$$\Rightarrow \quad y_1 \sim \frac{c_2^+}{x^{1/4}} e^{-\frac{2x^{3/2}}{3\varepsilon}}, \quad y_2 \sim \frac{c_2^-}{x^{1/4}} e^{+\frac{2x^{3/2}}{3\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

Valid for $u = O(1)$ as $\varepsilon \to 0$ ∴ lose validity when $x = O(\varepsilon^{2/3})$

Q4 $\varepsilon y'' + y' + xy = 0$ for $0 < x < 1$, with $y(0) = 0,\ y(1) = 1$.

(a) $y = e^{S(x)/\varepsilon} \Rightarrow y' = \frac{S'}{\varepsilon} e^{S/\varepsilon} \Rightarrow y'' = \left[\frac{(S')^2}{\varepsilon^2} + \frac{S''}{\varepsilon}\right] e^{S/\varepsilon}$

ODE $\Rightarrow \quad (S')^2 + S' + \varepsilon(S'' + x) = 0$

$S \sim S_0(x) + \varepsilon S_1(x) + \ldots$ as $\varepsilon \to 0^+ \Rightarrow$

$O(\varepsilon^0)$: $(S_0')^2 + S_0' = 0 \Rightarrow S_0' = 0, -1 \Rightarrow S_0 = A_1,\ B_1 - x$

$(A_1, B_1 \in \mathbb{R})$

$O(\varepsilon^1)$: $2S_0'S_1' + S_1' + S_0'' + x = 0$

$S_0 = A_1 \Rightarrow S_1' = -x \Rightarrow S_1 = A_2 - \frac{1}{2}x^2 \quad (A_2 \in \mathbb{R})$

$S_0 = B_1 - x \Rightarrow S_1' = x \Rightarrow S_1 = B_2 + \frac{1}{2}x^2 \quad (B_2 \in \mathbb{R})$

Hence, general solution $y \sim A_3 e^{-\frac{1}{2}x^2} + B_3 e^{-x/\varepsilon + \frac{1}{2}x^2}$, where $A_3, B_3 \in \mathbb{R}$ and we have absorbed $e^{A_1/\varepsilon + A_2}$ into A_3 and $e^{B_1/\varepsilon + B_2}$ into B_3.

$y(0) = 0 \Rightarrow A_3 \sim -B_3$

$y(1) = 1 \Rightarrow A_3 e^{-1/2} + B_3 e^{-1/\varepsilon + 1/2} \sim 1$

Thus, $A_3 \sim -B_3 \sim \dfrac{1}{e^{-1/2} - e^{-1/\varepsilon + 1/2}} = \dfrac{e^{1/2}}{1 - e^{1 - 1/\varepsilon}}$

giving $y \sim \dfrac{e^{(1-x^2)/2} - e^{-x/\varepsilon + (1+x^2)/2}}{1 - e^{1-1/\varepsilon}}$ as $\varepsilon \to 0^+$

(b) $x = 1 + \varepsilon X,\ y = Y(X) \Rightarrow \frac{d^2Y}{dX^2} + \frac{dY}{dX} + \varepsilon(1 + \varepsilon X)Y = 0$

$\Rightarrow Y \sim C_1 + C_2 e^{-X} \quad (C_1, C_2 \in \mathbb{R})$ as $\varepsilon \to 0^+$

Matching with outer ($1 - x = O(1)$) requires $Y(-\infty)$ to be finite $\Rightarrow C_2 = 0 \Rightarrow$ no BL at $x = 1$ at leading order.

<u>Outer:</u> $y \sim y_0(x) + \varepsilon y_1(x) + \dots$ as $\varepsilon \to 0^+$ with $x = O(1)$ $\Rightarrow$

$O(\varepsilon^0)$: $y_0' + x y_0 = 0$ for $0 < x < 1$, with $y_0(1) = 1$ ($\because$ no BL at $x = 1$)

$\Rightarrow \frac{y_0'}{y_0} = -x \Rightarrow \ln|y_0| = D_1 - \frac{x^2}{2} \Rightarrow y_0 = D_2 e^{-x^2/2}$ ($D_1, D_2 \in \mathbb{R}$, $|D_2| = e^{D_1}$)

$y_0(1) = 1 \Rightarrow 1 = D_2 e^{-1/2} \Rightarrow \underline{\underline{y_0(x) = e^{(1-x^2)/2}}}$

<u>Inner:</u> $x = \varepsilon X$, $y = Y(X) \Rightarrow \frac{d^2Y}{dX^2} + \frac{dY}{dX} + \varepsilon^2 X Y = 0$ (balancing 1st and 2nd terms)

$Y \sim Y_0(X) + \varepsilon Y_1(X) + \dots$ as $\varepsilon \to 0^+$ with $X = O(1)$ $\Rightarrow$

$O(\varepsilon^0)$: $\frac{d^2Y_0}{dX^2} + \frac{dY_0}{dX} = 0$ for $X > 0$, with $Y_0(0) = 0$ (BC)

$\Rightarrow \underline{\underline{Y_0 = E_1(1 - e^{-X}) \quad (E_1 \in \mathbb{R})}}$

<u>Matching:</u>

$(1.t.o.) = e^{(1-x^2)/2}$
$\Rightarrow (1.t.o.)$ in inner variables $= e^{(1-\varepsilon^2 X^2)/2} \sim e^{1/2}$
$\Rightarrow (1.t.i.)(1.t.o.) = e^{1/2}$

$(1.t.i.) = E_1(1 - e^{-X})$
$\Rightarrow (1.t.i.)$ in outer variables $= E_1(1 - e^{-x/\varepsilon}) \sim E_1$
$\Rightarrow (1.t.o.)(1.t.i.) = E_1$

$(1.t.i.)(1.t.o.) = (1.t.o.)(1.t.i.) \Rightarrow \underline{\underline{E_1 = e^{1/2}}}$

<u>Composite expansion:</u> Additive composite expansion given by

$$y \sim y_0(x) + Y_0(x/\varepsilon) - (1.t.i.)(1.t.o.)$$

$$\underline{\underline{= e^{(1-x^2)/2} - e^{1/2 - x/\varepsilon} \quad \text{as } \varepsilon \to 0^+}}$$

(because $(1.t.i.)(1.t.o.)$ counted twice in $y_0(x) + Y_0(x/\varepsilon)$).

Q5 $\varepsilon^2 y'' + (1-x)y = 0$ for $x > 0$, with $y(0) = 1$, $y(\infty) = 0$.

(a) Let $x = 1 + \varepsilon^{2/3} X$, $y = Y(X)$ $\Rightarrow$ $\dfrac{d^2Y}{dX^2} - XY = 0$ for $X > -\varepsilon^{-2/3}$

$\Rightarrow$ $Y(X) = a\,Ai(X) + b\,Bi(X)$ $\quad (a, b \in \mathbb{R})$

BCs become $Y(-\varepsilon^{-2/3}) = 1$, $Y(\infty) = 0$.

As $X \to \infty$, $Ai(X) \sim \dfrac{1}{2\sqrt{\pi}X^{1/4}} e^{-\frac{2}{3}X^{3/2}}$, $Bi(X) \sim \dfrac{1}{\sqrt{\pi}X^{1/4}} e^{\frac{2}{3}X^{3/2}}$

$Y(\infty) = 0 \Rightarrow b = 0$

$Y(-\varepsilon^{-2/3}) = 1 \Rightarrow a\,Ai(-\varepsilon^{-2/3}) = 1$

Hence, exact solution is $y(x) = Y(X) = \dfrac{Ai(X)}{Ai(-\varepsilon^{-2/3})} = \dfrac{Ai(\varepsilon^{-2/3}(x-1))}{Ai(-\varepsilon^{-2/3})}$

(b) $y = A(x)e^{i\phi(x)/\varepsilon}$ $\Rightarrow$ $y' = \left(\dfrac{iA\phi'}{\varepsilon} + A'\right)e^{i\phi/\varepsilon}$

$\Rightarrow$ $y'' = \left(\dfrac{-A(\phi')^2}{\varepsilon^2} + \dfrac{2iA'\phi'}{\varepsilon} + \dfrac{iA\phi''}{\varepsilon} + A''\right)e^{i\phi/\varepsilon}$

ODE $\Rightarrow$ $-A(\phi')^2 + \varepsilon(2iA'\phi' + iA\phi'') + \varepsilon^2 A'' + (1-x)A = 0$

$A \sim A_0(x) + \varepsilon A_1(x) + \dots$ as $\varepsilon \to 0$ $\Rightarrow$

$O(\varepsilon^0)$: $\quad -A_0(\phi')^2 + (1-x)A_0 \Rightarrow \phi' = \pm(1-x)^{1/2} \Rightarrow \phi = \pm\frac{2}{3}(1-x)^{\frac{3}{2}} + \text{const}$

$O(\varepsilon^1)$: $-A_1(\phi')^2 + 2iA_0'\phi' + A_0\phi'' + (1-x)A_1 = 0 \Rightarrow (A_0^2\phi')' = 0$

$\Rightarrow$ $A_0^2 = \dfrac{\text{constant}}{\phi'}$ $\Rightarrow$ $A_0 = \dfrac{\text{constant}}{(1-x)^{1/4}}$

Hence, character of solution changes depending on whether $x > 1$ or $x < 1$

RH outer $x > 1$

$y(\infty) = 0 \Rightarrow$ need to eliminate growing solution, giving

$$y \sim \frac{c_1}{(x-1)^{1/4}} \exp\left(-\frac{2}{3\varepsilon}(x-1)^{3/2}\right) \text{ as } \varepsilon \to 0^+ \text{ with } x > 1,\ x-1 = \text{ord}(1)$$

where $c_1 \in \mathbb{R}$

LH outer $0 < x < 1$

Now both $\phi = \pm\frac{2}{3}(1-x)^{3/2}$ are admissible, giving

$$y \sim \frac{c_2}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2} + \alpha_1\right) \text{ as } \varepsilon \to 0^+ \text{ with } 0 < x < 1,\ x = \text{ord}(1)$$

where $c_2, \alpha_1 \in \mathbb{R}$

$$y(0) = 1 \Rightarrow c_2 \Rightarrow y \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon} + \alpha_1\right)}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2} + \alpha_1\right)$$

Inner region near $x = 1$

Outer solutions unbounded as $x \to 1^{\pm}$, so seek an inner solution by scaling $x = 1 + \delta(\varepsilon)X$, $y = \delta(\varepsilon)^{-1/4} Y(X)$, giving Airy's equation $\frac{d^2Y}{dX^2} = XY$ provided $\delta(\varepsilon) = \varepsilon^{2/3}$.

General solution is

$$Y(X) = c_3 \text{Ai}(X) + c_4 \text{Bi}(X) \qquad (c_3, c_4 \in \mathbb{R})$$

Matching inner ($X \to \infty$) with RH outer ($x \to 1^+$)

Safer to use intermediate variable $\hat{x}$ to match:

$$x - 1 = \varepsilon^{\alpha}\hat{x} = \varepsilon^{2/3}X \qquad (0 < \alpha < \tfrac{2}{3}).$$

- $X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to \infty$ as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$\delta^{-1/4} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) = \frac{C_3}{\varepsilon^{1/6}} \mathrm{Ai}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) + \frac{C_4}{\varepsilon^{1/6}} \mathrm{Bi}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)$$

$$\sim \frac{C_3}{\varepsilon^{1/6}} \frac{1}{2\sqrt{\pi}(\hat{x}/\varepsilon^{2/3-\alpha})^{1/4}} \exp\left(-\frac{2}{3}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2}\right)$$

$$+ \frac{C_4}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}(\hat{x}/\varepsilon^{2/3-\alpha})^{1/4}} \exp\left(\frac{2}{3}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2}\right)$$

- $x = 1 + \varepsilon^{\alpha}\hat{x} \to 1^+$ as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$y(1+\varepsilon^{\alpha}\hat{x}) \sim \frac{C_1}{(\varepsilon^{\alpha}\hat{x})^{1/4}} \exp\left(-\frac{2}{3\varepsilon}(\varepsilon^{\alpha}\hat{x})^{3/2}\right)$$

- Matching $\Rightarrow$ $\underline{\underline{C_4 = 0, \quad C_1 = \frac{C_3}{2\sqrt{\pi}}}}$

Matching inner ($X \to -\infty$) with LH outer ($x \to 1^-$)

- $X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to -\infty$ as $\varepsilon \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$\delta^{-1/4} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) - \frac{C_3}{\varepsilon^{1/6}} \mathrm{Ai}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) \sim \frac{C_3}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}(-\hat{x}/\varepsilon^{2/3-\alpha})^{1/4}} \sin\left(\frac{2}{3}\left(\frac{-\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2} + \frac{\pi}{4}\right)$$

- $x = 1 + \varepsilon^{\alpha}\hat{x} \to 1^-$ as $\varepsilon \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \mathrm{ord}(1) \Rightarrow$

$$y(1+\varepsilon^{\alpha}\hat{x}) \sim \frac{\mathrm{cosec}\left(\frac{2}{3\varepsilon} + \alpha_1\right)}{(-\varepsilon^{\alpha}\hat{x})^{1/4}} \sin\left(\frac{2}{3\varepsilon}(-\varepsilon^{\alpha}\hat{x})^{3/2} + \alpha_1\right)$$

- Matching $\Rightarrow$ $\underline{\underline{\alpha_1 = \frac{\pi}{4} \text{ (wlog)}, \quad \frac{C_3}{\sqrt{\pi}} = \mathrm{cosec}\left(\frac{2}{3\varepsilon} + \frac{\pi}{4}\right)}}$

- Hence, $C_1 = \frac{1}{2}\mathrm{cosec}\left(\frac{2}{3\varepsilon} + \frac{\pi}{4}\right)$ and we're done.

- NB: plots show excellent agreement with exact solution for $\varepsilon \lesssim 0.1$

Q6 $$I(\varepsilon) = \frac{e^{1/\varepsilon}}{\varepsilon}\int_{1/\varepsilon}^{\infty} \underbrace{\frac{1}{t}}_{f} \underbrace{e^{-t}}_{g'} \, dt$$

$$= \frac{e^{1/\varepsilon}}{\varepsilon}\left[\underbrace{\frac{1}{t}}_{f} \underbrace{-e^{-t}}_{g} \Big|_{1/\varepsilon}^{\infty} - \int_{1/\varepsilon}^{\infty} \underbrace{\frac{-1}{t^2}}_{f'} \underbrace{-e^{-t}}_{g} \, dt \right]$$

$$= \frac{e^{1/\varepsilon}}{\varepsilon}\left[e^{-1/\varepsilon}\varepsilon - \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^2}\, dt \right]$$

$\Rightarrow$ true for $N=1$

Assume true for N, then inductive step is

$$(-1)^N N! \int_{1/\varepsilon}^{\infty} \underbrace{\frac{1}{t^{N+1}}}_{f} \underbrace{e^{-t}}_{g'} \, dt = (-1)^N N! \left[\underbrace{\frac{1}{t^{N+1}}}_{f} \underbrace{-e^{-t}}_{g} \Big|_{1/\varepsilon}^{\infty} - \int_{1/\varepsilon}^{\infty} \underbrace{\frac{-(N+1)}{t^{N+2}}}_{f'} \underbrace{-e^{-t}}_{g} \, dt \right]$$

$$= (-1)^N N! \, e^{-1/\varepsilon} \varepsilon^{N+1} + (-1)^{N+1}(N+1)! \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N+2}}\, dt$$

$\Rightarrow$ true for $N+1$, so true $\forall n \in \mathbb{N}$ by induction. □

(b) $$\left| \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N+1}}\, dt \right| < \varepsilon^{N+1} \int_{1/\varepsilon}^{\infty} e^{-t}\, dt = e^{-1/\varepsilon}\varepsilon^{N+1}$$

$$\Rightarrow \left| I(\varepsilon) - \sum_{n=0}^{N-1} (-1)^n n! \varepsilon^n \right| = \left| (-1)^N N! \frac{e^{1/\varepsilon}}{\varepsilon} \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N+1}}\, dt \right|$$

$$< N! \, \varepsilon^N$$

$$\Rightarrow \frac{\left| I(\varepsilon) - \sum_{n=0}^{N-1} (-1)^n n! \varepsilon^n \right|}{\left| (-1)^{N-1}(N-1)! \varepsilon^{N-1} \right|} < N\varepsilon \to 0 \text{ as } \varepsilon \to 0 \ \forall N \in \mathbb{N}$$

$$\Rightarrow I(\varepsilon) \sim \sum_{n=0}^{\infty} (-1)^n n! \varepsilon^n$$

□

(c) $S_N(\varepsilon) = \sum_{n=0}^{N-1} (-1)^n n! \varepsilon^n \to \infty$ as $N \to \infty$ $\forall_{\varepsilon > 0}$.

$|I(0.2) - S_N(0.2)|$ minimal for $N = 5$.

$|I(0.1) - S_N(0.1)|$ minimal for $N = 10$.

See plots over page.

Note that these values correspond to truncating at the smallest term $a_n(\varepsilon) = (-1)^n n! \varepsilon^n$ (called optimal truncation) because the ratio of successive terms $|a_n(\varepsilon) / a_{n-1}(\varepsilon)| = n\varepsilon$ begins to grow when $n > 1/\varepsilon$.

Given $0 < \varepsilon \ll 1$, optimal truncation truncates at $N(\varepsilon)$, where $N(\varepsilon)\varepsilon \leq 1 < (N(\varepsilon)+1)\varepsilon$.

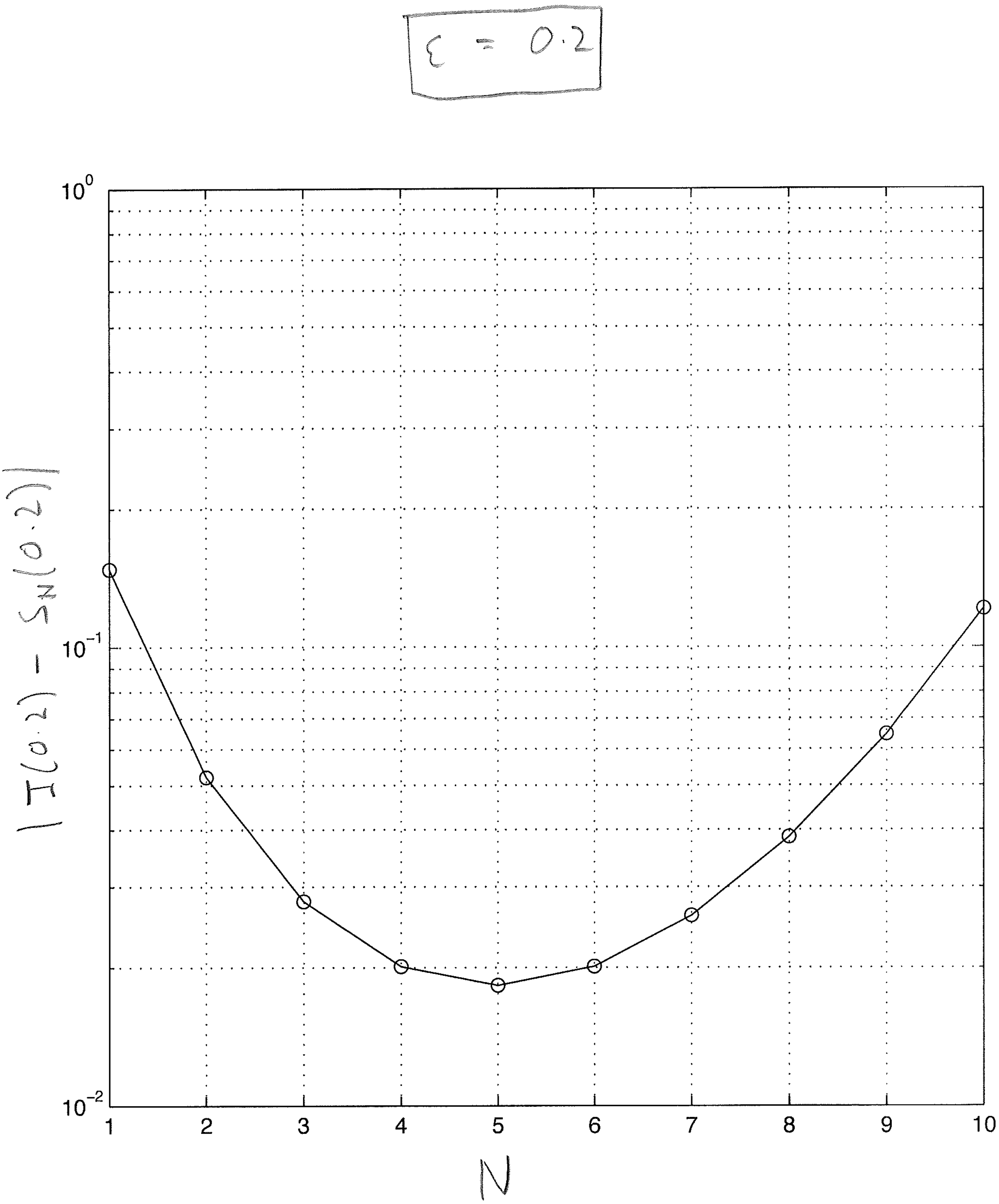
Remainder

$$R_{N(\varepsilon)}(\varepsilon) = \left| (-1)^{N(\varepsilon)} N(\varepsilon)! \frac{e^{1/\varepsilon}}{\varepsilon} \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N(\varepsilon)+1}} \, dt \right|$$

$$\sim \sqrt{\frac{\pi}{2\varepsilon}} \, e^{-1/\varepsilon} \quad \text{as } \varepsilon \to 0^+$$

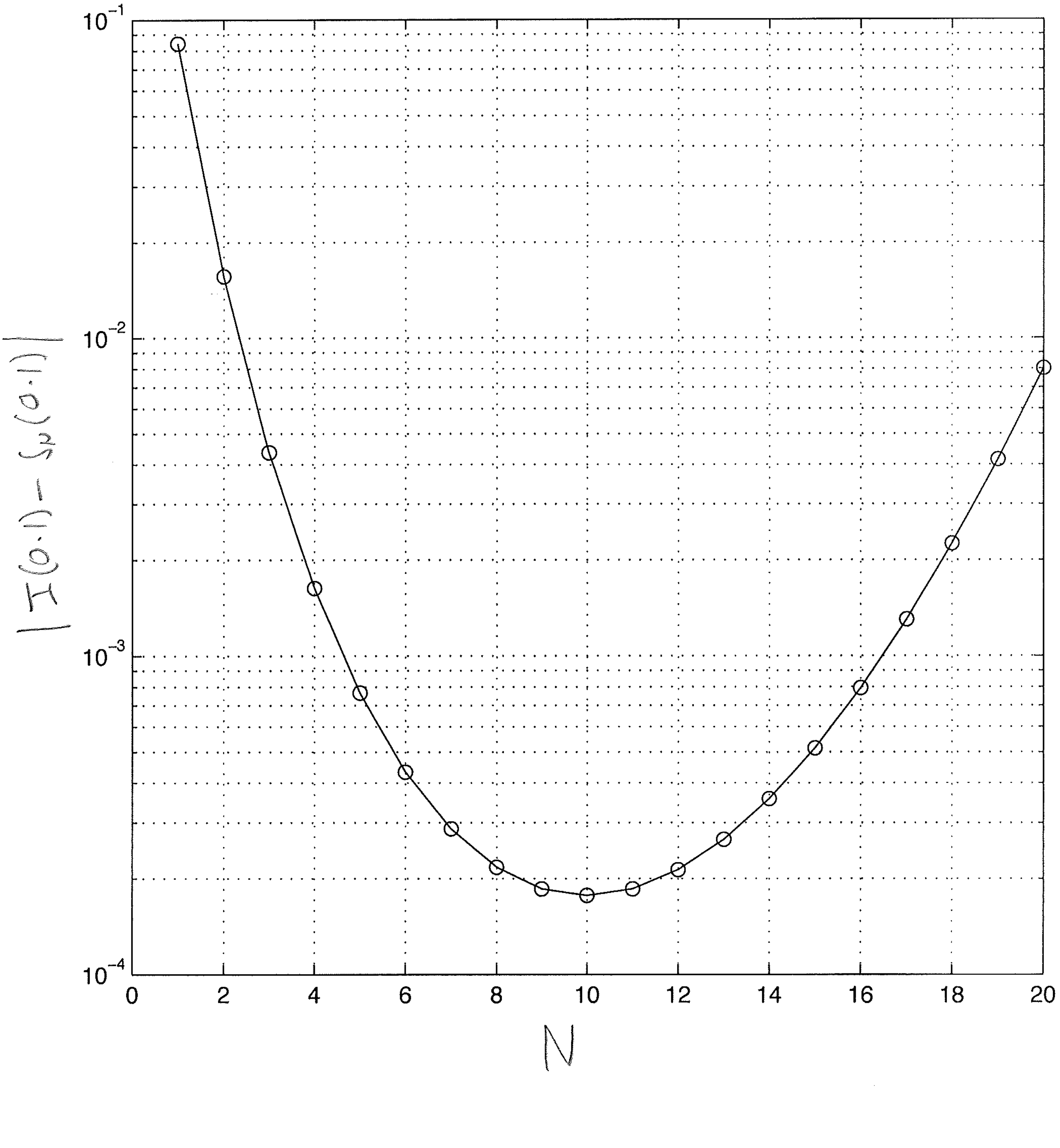
via Laplace's method (see sheet 2)

$\Rightarrow$ error exponentially small with optimal truncation!



ε = 0.2
10^0
10^-1
10^-2
1
2
3
4
5
6
7
8
9
10
|I(0.2) − S_N(0.2)|
N

$\varepsilon = 0.1$



Q7 $\varepsilon \nabla^2 u = u$ in $r < 1$, with $u = 1$ on $r = 1$.

Outer : $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $1-r = O(1)$.

$O(\varepsilon^0)$: $u_0 = 0$

$O(\varepsilon^1)$: $u_1 = \nabla^2 u_0 = 0$

$O(\varepsilon^2)$: $u_2 = \nabla^2 u_1 = 0$

Induction $\Rightarrow$ $u = O(\varepsilon^n)$ $\forall n \in \mathbb{N}$ as $\varepsilon \to 0^+$ □

Inner : $u(r,\theta) = U(R,\theta)$, $r = 1 - \delta(\varepsilon)R$, with $\delta \to 0$, $R = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2} U_{RR} - \frac{\varepsilon}{\delta(1-\delta R)} U_R + \frac{\varepsilon}{(1-\delta R)^2} U_{\theta\theta} - U = 0$$

Balance 1st and 4th term by setting $\delta = \varepsilon^{1/2}$ to obtain

$$U_{RR} - \frac{\varepsilon^{1/2}}{(1-\varepsilon^{1/2}R)} U_R + \frac{\varepsilon}{(1-\varepsilon^{1/2}R)^2} U_{\theta\theta} - U = 0$$

$U \sim U_0(R,\theta) + \varepsilon^{1/2} U_1(R,\theta) + \dots$ as $\varepsilon \to 0^+$ with $R = O(1)$.

$O(\varepsilon^0)$: $U_{0RR} - U_0 = 0$ in $R > 0$, with $U_0 = 1$ on $R = 0$

$$\Rightarrow \quad U_0 = A e^{R} + (1-A) e^{-R} \quad (A \in \mathbb{R})$$

Matching : $(\text{l.t.o.}) = 0$ $\Rightarrow$ $(\text{l.t.i.})(\text{l.t.o.}) = 0$

$\Rightarrow$ $(\text{l.t.o.})(\text{l.t.i.}) = 0$

$\Rightarrow$ $U_0 \to 0$ as $R \to \infty$

$\Rightarrow$ $A = 0$

Hence, $u = e^{-R} + O(\varepsilon^{1/2})$ as $\varepsilon \to 0^+$ with $\varepsilon^{1/2}(1-r) = R = O(1)$ □

Given exact solution $u = I_0(r/\sqrt{\varepsilon}) / I_0(1/\sqrt{\varepsilon})$, where

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cos(ix\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^\pi e^{-i(ix\sin\theta)} + e^{i(ix\sin\theta)}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^\pi e^{x\sin\theta} + e^{-x\sin\theta}\,d\theta$$

$$\sim \frac{1}{2\pi}\int_0^\pi e^{x\sin\theta}\,d\theta$$ as $x \to \infty$ ∵ 1st integral is exponentially dominant ∵ $\sin\theta > 0$ for $0 < \theta < \pi$.

$$\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x[1-\frac{1}{2}(\theta-\frac{\pi}{2})^2+\dots]}\,d\theta$$ via Laplace's method ∵ $\phi(\theta) = \sin\theta$ has max at $\theta = \pi/2$.

$$\sim \frac{e^x}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{xs^2}{2}}\,ds \qquad (\theta - \tfrac{\pi}{2} = s)$$

$$= \frac{e^x}{\sqrt{2\pi}}\sqrt{\frac{2}{x}}\underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}_{=\sqrt{\pi}} \qquad \left(s = \sqrt{\tfrac{2}{x}}\,t\right)$$

$$\Rightarrow I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty.$$

Thus,

$$u \sim \frac{1}{\sqrt{r}}\, e^{-(1-r)/\sqrt{\varepsilon}}$$ as $\varepsilon \to 0^+$ with $r = O(1)$, $1-r = O(1)$;

$$u \sim \frac{\sqrt{2\pi}\,e^{-1/\sqrt{\varepsilon}}}{\varepsilon^{1/4}}\, I_0(\rho)$$ as $\varepsilon \to 0^+$ with $\rho = \varepsilon^{-1/2} r = O(1)$;

$$u \sim \frac{1}{\sqrt{1-\varepsilon^{1/2}R}}\, e^{-R} = e^{-R} + O(\varepsilon^{1/2})$$ as $\varepsilon \to 0^+$ with $R = \varepsilon^{-1/2}(1-r) = O(1)$,

in agreement with formal BL theory. □

(b) $\varepsilon(u_{xx} + u_{yy}) = u_x$ in $y > 0$;
$u = 1$ on $y = 0, x > 0$;
$u_y = 0$ on $y = 0, x < 0$;
$u \to 0$ as $r \to \infty$.

Outer: $u \sim u_0 + \varepsilon u_1 + \ldots$ as $\varepsilon \to 0^+$ with $x, y = O(1)$

$O(\varepsilon^0)$: $u_{0x} = 0$ with $u_0 = 0$ at ∞ $\Rightarrow u_0 = 0$
$O(\varepsilon^1)$: $u_{1x} = 0$ with $u_1 = 0$ at ∞ $\Rightarrow u_1 = 0$

Induction $\Rightarrow$ $u = O(\varepsilon^n)$ $\forall n \in \mathbb{N}$ as $\varepsilon \to 0^+$ with $x, y = O(1)$ □

Inner: $u(x,y) = U(x,Y)$, $y = \delta(\varepsilon)Y$, with $\delta \to 0$, $Y = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \varepsilon U_{xx} + \frac{\varepsilon}{\delta^2} U_{YY} - U_x = 0$$

Balance 2nd and 3rd terms by setting $\delta = \varepsilon^{1/2}$ to obtain

$$\varepsilon U_{xx} + U_{YY} - U_x = 0$$

$U \sim U_0(x,Y) + \varepsilon U_1(x,Y) + \ldots$ as $\varepsilon \to 0^+$ with $Y = O(1)$.

$O(\varepsilon^0)$: $U_{0YY} - U_{0x} = 0$ in $Y > 0, x > 0$.

BC: $U_0(x, 0) = 1$ for $x > 0$.

Matching: $(\text{l.t.o.}) = 0 \Rightarrow (\text{l.t.i.})(\text{l.t.o.}) = 0$
$\Rightarrow (\text{l.t.o.})(\text{l.t.i.}) = 0$
$\Rightarrow U_0 \to 0$ as $Y \to \infty$ for $x > 0$.

Seek similarity solution $U_0 = f(\eta)$, $\eta = Y/\sqrt{x}$

$\Rightarrow \quad \eta_x = -\frac{\eta}{2x}$

$\eta_Y = \frac{1}{x^{1/2}}$

$U_{0x} = f'(\eta)\eta_x = -\frac{\eta f'(\eta)}{2x}$

$U_{0YY} = f''(\eta)(\eta_Y)^2 = \frac{f''(\eta)}{x}$

PDE $\Rightarrow \quad f'' + \frac{1}{2}\eta f' = 0 \quad$ for $\eta > 0$.

BCs : $U_0 = 1$ on $Y = 0, x > 0 \quad \Rightarrow \quad f(0) = 1$
$U_0 \to 0$ as $Y \to \infty, x > 0 \quad \Rightarrow \quad f(\infty) = 0$

Now, $\frac{f''}{f'} = -\frac{\eta}{2} \quad \Rightarrow \quad \ln|f'| = c_1 - \frac{\eta^2}{4} \quad (c_1 \in \mathbb{R})$

$\Rightarrow \quad f' = c_2 e^{-\eta^2/4} \quad (|c_2| = e^{c_1})$

$\Rightarrow \quad f(\eta) = c_1 - c_2 \int_{\eta}^{\infty} e^{-s^2/4}\,ds$

$= c_1 - 2c_2 \int_{\eta/2}^{\infty} e^{-t^2}\,dt \quad (s = 2t)$

$= c_1 - 2c_2 \,\mathrm{erfc}\left(\frac{\eta}{2}\right)$

$f(\infty) = 0 \quad \Rightarrow \quad 0 = c_1 - 0$
$f(0) = 1 \quad \Rightarrow \quad 1 = -2c_2\,\mathrm{erfc}(0) = -2c_2$

Hence, $f(\eta) = \mathrm{erfc}\left(\frac{\eta}{2}\right) \Rightarrow U = \mathrm{erfc}\left(\frac{Y}{2\sqrt{x}}\right) + O(\varepsilon)$ as $\varepsilon \to 0^+$
with $Y = \varepsilon^{-1/2} y = O(1)$
and $x = O(1)$ □

Neither approximation holds for $X = \varepsilon^{-1}x = O(1), Y = \varepsilon^{-1}y = O(1)$
$\Rightarrow u_{XX} + u_{YY} = u_X$ in $Y > 0$.

Q8) $\ddot{x} + \varepsilon(x^2-\lambda)\dot{x} + x = 0 \Rightarrow$ same as Q1a until

$O(\varepsilon^1)$: $x_{1tt} + x_1 = -2x_{0tT} - (x_0^2-\lambda)x_{0t}$

$= -(iA_T e^{it} - i\bar{A}_T e^{-it})$

$-(\frac{1}{4}(Ae^{it} - \bar{A}e^{-it})^2 - \lambda)(iAe^{it} - i\bar{A}e^{-it})\frac{1}{2}$

$= \left[-iA_T - \frac{1}{4}A^2\left(\frac{-i\bar{A}}{2}\right) - \left(\frac{2}{4}A\bar{A}-\lambda\right)\frac{iA}{2}\right]e^{it}$

$+$ c.c. $+$ non-secular

Can suppress secular terms only if $-2iA_T + \frac{i}{4}A^2\bar{A} - i(\frac{1}{2}A\bar{A}-\lambda)A = 0$

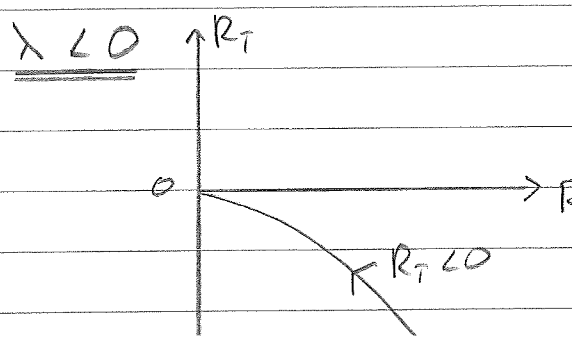
$$\Rightarrow 2A_T = \left(\lambda - \frac{|A|^2}{4}\right)A$$

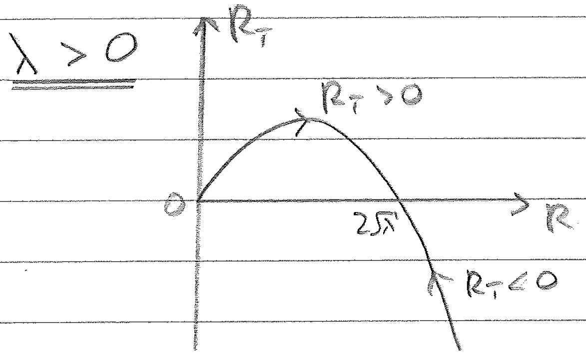
$A = Re^{i\Theta} \Rightarrow 2(R_T + iR\Theta_T) = (\lambda - \frac{R^2}{4})R$

$\Rightarrow \Theta_T = 0,\ 2R_T = (\lambda - \frac{R^2}{4})R$

$\Rightarrow \Theta(T) = \Theta(0),\quad R(T) \to \begin{cases} 0, & \lambda < 0 \\ 2\sqrt{\lambda}, & \lambda > 0 \end{cases}$

because $R_T \gtrless 0 \Leftrightarrow \lambda - \frac{R^2}{4} \gtrless 0$ for $R > 0$:





Thus, for $\lambda < 0$, solution tends to only steady solution $R = 0$, while for $\lambda > 0$, solution tends to a periodic orbit with period 2π and amplitude $2\sqrt{\lambda}$ at leading order ($\because x_0 = R(T)\cos(t + \Theta(0))$). This is called a Hopf bifurcation.

□