C5.5
Perturbation Methods

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1. Introduction

. Perturbation methods exploit a small, or large, parameter to make systematic, precise approximations.

{ Difficult to give rules for perturbation methods, only guidelines.

· Hinch, Bender & Orzag, Supplementary Notes online.

2. Algebraic Equations

Example $\quad x^2 + \varepsilon x - 1 = 0, \qquad |\varepsilon| \ll 1.$

$$x = \left[-\frac{\varepsilon}{2} \pm \sqrt{1 + \left(\frac{\varepsilon}{2}\right)^2} \right] \overset{\text{binomial expansion}}{=} \begin{cases} 1 - \varepsilon/2 + \varepsilon^2/8 + \cdots \\ -1 - \varepsilon/2 - \varepsilon^2/8 + \cdots \end{cases}$$

$\underbrace{\text{for } |\varepsilon/2| < 1.}_{\text{for convergence}}$

In addition to convergence, truncated expansions give good approximations to the roots when $|\varepsilon| \ll 1$.

For $\varepsilon = 0.1$ and the positive root

$x \doteq$	1	1st term	exact root is 0.95124922...
	0.95	2nd term	
	0.95125	3rd term	
	0.951249	4th term	

Solved, then approximated .... usually we approximate, then solve

2.1 Iterative method

$$x^2 + \varepsilon x - 1 = 0$$

For positive root $x = \sqrt{1 - \varepsilon x}$ by rearrangement.

Consider iteration $x_{n+1} = g_\varepsilon(x_n) := \sqrt{1 - \varepsilon x_n}$

Note, if x^* is a root, so that $x_* = g_\varepsilon(x_*)$ then if $|x_n - x^*|$ is small,

$$x_{n+1} - x^* = g_\varepsilon(x_n) - x^* = g_\varepsilon(x^* + (x_n - x^*)) - x^*$$

$$= \underbrace{(g_\varepsilon(x^*) - x^*)}_{0} + (x_n - x^*) g_\varepsilon'(x^*) + \dots$$

Also $g_\varepsilon'(x^*) = \dfrac{-\varepsilon/2}{\sqrt{1 - \varepsilon x^*}} \triangleq -\varepsilon/2 \quad \therefore |x_{n+1} - x^*| \triangleq \left|\dfrac{\varepsilon}{2}\right| |x_n - x^*|$

Hence iteration converges.

Beginning with $x_0 = 1$, $x_1 = \sqrt{1 - \varepsilon} = \underbrace{1 - \varepsilon/2}_{\text{Correct}} - \varepsilon^2/8 - \varepsilon^3/16 + \dots$

$$x_2 = \sqrt{1 - \varepsilon\left(1 - \frac{\varepsilon}{2} + \dots\right)}$$

$$= 1 - \frac{\varepsilon}{2}\left(1 - \varepsilon/2 + \dots\right) - \frac{\varepsilon^2}{8}\left(1 - \varepsilon/2 + \dots\right)^2 - \frac{\varepsilon^3}{16}\left(1 - \frac{\varepsilon}{2} + \dots\right)^3$$

$$= \underbrace{1 - \varepsilon/2 + \varepsilon^2/8}_{\text{Correct}} + \underbrace{\dots}_{\text{Not correct}}$$

At each iteration, more terms correct, but more work required

If solution not known, can only confirm terms are correct by performing a further iteration and checking they do not change.

For fast convergence, ideally want $g_\varepsilon(x)$ such that $g'_\varepsilon(x) \to 0$ as $\varepsilon \to 0$.

2.2 Expansion Method $\boxed{\text{(Much more common)}}$

For $\varepsilon = 0$, $x = \pm 1$.

to be determined
no dependence on ε

Positive root Let $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

$$(1+\varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^2 + \varepsilon(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) - 1 = 0$$

Term involving ε^0

$$1 - 1 = 0$$

ε^1 $\quad 2x_1 + 1 = 0 \quad \therefore x_1 = -\tfrac{1}{2}$

ε^2 $\quad 2x_2 + x_1^2 + x_1 = 0 \quad \therefore x_2 = \tfrac{1}{8}$ etc.

set coefficients of powers of ε to zero for each power... ~~assume~~ as this is to hold for all sufficiently small ε.

Caveat Must know/assume form of expansion

2.3 Singular Perturbations

$$\varepsilon x^2 + x - 1 = 0 \qquad |\varepsilon| \ll 1$$

$\varepsilon = 0$, one root $x = 1$. $\quad \varepsilon \neq 0$ two roots.

Singular ... the case with $\varepsilon = 0$ differs in an important way from the case with $\varepsilon \to 0$.

Non-singular problems are *regular*.

Solve

$$x = \frac{1}{2\varepsilon}\left[-1 \pm \sqrt{1+4\varepsilon}\right]$$

$$= \begin{cases} 1-\varepsilon+2\varepsilon^2+\dots \\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+\dots \end{cases}$$

for $|4\varepsilon|<1$ by binomial expansion

Second root blows up as $\varepsilon \to 0$.

Iterative method

$g_\varepsilon(x) = 1-\varepsilon x^2$ for 1^{st} root

$g_\varepsilon(x) = \dfrac{1-x}{\varepsilon x}$ for 2^{nd} root

Both derivatives are small near their respective roots and tend to zero as $\varepsilon \to 0$.

Expansion method (2^{nd} root)

Let $x = \dfrac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and consider $\varepsilon x^2 + x - 1 = 0$

At ε^{-1} $\quad x_{-1}^2 + x_{-1} = 0$

At ε^0 $\quad 2x_{-1}x_0 + x_0 - 1 = 0$

At ε^1 $\quad (2x_{-1}x_1 + x_0^2) + x_1 = 0$

$x_{-1} = 0, -1$

$x_0 = 1, -1$

$x_1 = -1, 1$

regular root $\quad$ 2nd root

Rescaling

Let $x = X/\varepsilon \Rightarrow X^2 + X - \varepsilon = 0$, regular.

Finding correct starting point for expansion same as finding a rescaling that makes problem regular.

rescaling that makes problem regular

2.4 Finding the correct rescaling

Systematic approach $\qquad x = \delta(\varepsilon) X$ with X strictly order 1 as $\varepsilon \to 0$

We have

$$\underset{1}{\varepsilon\delta^2 X^2} + \underset{2}{\delta X} - \underset{3}{1} = 0$$

Vary δ from very small to large to identify dominant balances, where at least 2 terms are of the same order of magnitude, with all other terms smaller.

Scalings that yield dominant balances are distinguished limits.

$\delta \ll 1$	$1 \ll 2 \ll 3$	No balance
$\delta = 1$	$1 \ll 2 \sim 3$ same order of magnitude	Balance, regular root $X = 1 + \text{small}$
$1 \ll \delta \ll 1/\varepsilon$	$1 \ll 2 \gg 3$	No balance
$\delta = 1/\varepsilon$	$1 \sim 2 \gg 3$	Balance, singular root $x = 1/\varepsilon + \ldots$ $X = -1 + \text{small}$
$\delta \gg 1/\varepsilon$	$1 \gg 2 \gg 3$	No balance

$\therefore$ Distinguished limits are $\delta = 1,\ 1/\varepsilon$.

Alternative approach: Pairwise comparison

1 ~ 2 $\quad \varepsilon\delta^2 \sim \delta$ i.e. $\delta \sim 1/\varepsilon$ and 1 ~ 2 ≫ 3 ∴ Singular root.

1 ~ 3 $\quad \varepsilon\delta^2 \sim 1$ i.e. $\delta \sim 1/\sqrt{\varepsilon}$ and 1 ~ 3 ≪ 2 ∴ No dominant balance

2 ~ 3 $\quad \delta \sim 1$ $\quad$ 2 ~ 3 ≫ 1 ∴ Regular root.

2.5 Non-integer powers

$$(1-\varepsilon)x^2 - 2x + 1 = 0 \qquad |\varepsilon| \ll 1$$

$$x = \frac{1 \pm \sqrt{\varepsilon}}{1-\varepsilon} = 1 \pm \sqrt{\varepsilon} + \varepsilon \pm \varepsilon^{3/2} + \ldots$$

double root ... sign of danger

With $\varepsilon = 0$ $\quad (x-1)^2 = 0 \quad \therefore \; x = 1.$

Try $\quad x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

Know this will go wrong

Objective is to see how it goes wrong so we know what to do if we see analogous behaviour in the expansion method when we do not know the solution

At ε^0 $\qquad 1 - 2 + 1 = 0$

At ε^1 $\qquad -1 + 2x_1 - 2x_1 = 0$ $\qquad$ No solution, "unless x_1 blows up in some sense".

Try $\quad x = 1 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + \varepsilon^{3/2} x_{3/2} + \ldots$

At ε^0 $\qquad 1 - 2 + 1 = 0$

At $\varepsilon^{1/2}$ $\qquad 2x_{1/2} - 2x_{1/2} = 0$

At ε $\qquad 2x_1 + x_{1/2}^2 - 1 - 2x_1 = x_{1/2}^2 - 1 = 0$

$$\therefore x_{1/2} = \pm 1$$

etc.

2.6 Finding the correct expansion sequence

Let $x = 1 + \delta_1(\varepsilon) x_1$ where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, with x_1 strictly order one.

root when $\varepsilon = 0$

Prior to further expansion, x_1 retains an ε dependence; don't expand yet to keep working relatively simple

$$(1-\varepsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0$$

$$\cancel{1} + \cancel{2\delta_1 x_1} + \delta_1^2 x_1^2 - \varepsilon(1 + 2\delta_1 x_1 + \delta_1^2 x_1^2) - \cancel{2} - \cancel{2\delta_1 x_1} + \cancel{1} = 0$$

$$\underset{1}{\delta_1^2 x_1^2} - \underset{2}{\varepsilon} - \underset{3}{2\varepsilon\delta_1 x_1} - \underset{4}{\varepsilon\delta_1^2 x_1^2} = 0 \qquad (*)$$

Seek dominant balance 4 ≪ 1 always 3 ≪ 2 always

∴ 4 3 not in dominant balance ∴ 1 ~ 2 ∴ $\underline{\underline{\varepsilon \sim \delta_1^2}}$

∴ Let $\delta_1 = \sqrt{\varepsilon}$

further expansion here; now x_2 independent of ε as there are further terms

$$\therefore \quad x = 1 + \varepsilon^{1/2} \bar{x}_1 + \delta_2(\varepsilon) x_2 + \cdots \quad \text{with } \delta_2(\varepsilon) \ll \delta_1 = \varepsilon^{1/2}$$

From (*) we get at $O(\varepsilon)$

$$\bar{x}_1^2 - 1 = 0 \quad \therefore \underline{\underline{\bar{x}_1 = \pm 1}}$$

With $x \simeq 1 + \varepsilon^{1/2}$

$$x = 1 + \varepsilon^{1/2} + \delta_2(\varepsilon) x_2 + \cdots$$

$$(1-\varepsilon)(1 + \varepsilon^{1/2} + \delta_2 x_2 + \cdots)^2 - 2(1 + \varepsilon^{1/2} + \delta_2 x_2 + \cdots) + 1 = 0$$

Lots of algebra retaining all terms

$$\underset{1}{2\varepsilon^{1/2}\delta_2 x_2} + \underset{2}{\delta_2^2 x_2^2} - \underset{3}{2\varepsilon^{3/2}} - \underset{4}{\varepsilon^2} - \underset{5}{2\varepsilon\delta_2 x_2} - \underset{6}{2\varepsilon^{3/2}\delta_2 x_2} - \underset{7}{\varepsilon\delta_2^2 x_2^2} + \cdots = 0$$

Want dominant terms, but only know $\delta_2 \ll \varepsilon^{1/2}$, x_2 ~ order one

4 $\ll$ 3 for terms with no δ_2.

For terms with δ_2,

$$① \gg ⑦, \quad ① \sim \varepsilon^{1/2}\delta_2 \quad ⑦ \sim \varepsilon\delta_2^2 = (\varepsilon^{1/2}\delta_2)(\varepsilon^{1/2}\delta_2)$$
$$\sim ① \, \varepsilon^{1/2}\delta_2 \ll ①$$

Similarly for all other terms

$\therefore$ Dominant balance is between 1 and 3

$$2\varepsilon^{1/2}\delta_2 x_2 - 2\varepsilon^{3/2} = 0$$
$$\therefore \quad \delta_2 x_2 = \varepsilon$$

$x = 1 + \varepsilon^{1/2} + \delta_2(\varepsilon)x_2 + \cdots \quad \therefore \quad x = 1 + \varepsilon^{1/2} + \varepsilon + \ldots$

2.7 Iterative Method (again)

- Useful when expansion form not known

$$(1-\varepsilon)x^2 - 2x + 1 = 0$$
$$(x-1)^2 = \varepsilon x^2$$

For root > 1

$$x_{n+1} = g_\varepsilon(x_n) = 1 + \varepsilon^{1/2} x_n$$

Note $g'_\varepsilon(x_n) \to 0$ as $\varepsilon \to 0$

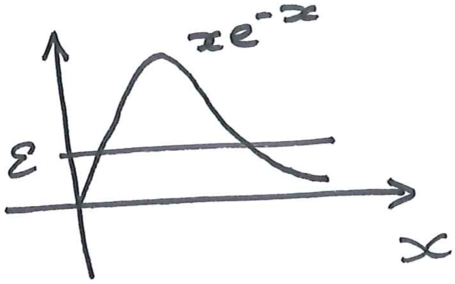
$\therefore x_0 = 1$ $\quad x_1 = 1 + \varepsilon^{1/2}, \quad x_2 = \ldots$ etc

solution if $\varepsilon = 0$

generates sequence

2.8 Logarithms

Example $\quad xe^{-x} = \varepsilon \qquad 0 < \varepsilon \ll 1$



Root near $x=0$ easy to find.

$$\left[\text{Let } \underbrace{x = 0}_{\substack{\text{solution}\\ \text{if } \varepsilon=0}} + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\dots \right]$$

Taylor expand xe^{-x} about 0 generate powers of $\varepsilon x_1 \dots$ which balance $\varepsilon \dots$ higher terms integer powers .. hence form of sequence

Other root $\to \infty$ as $\varepsilon \to 0$; expansion sequence not obvious

Take logs

$$\underset{①}{x} - \underset{②}{\log x} - \underset{③}{\log 1/\varepsilon} = 0$$

For x large $|①| \gg |②|$ $\therefore$ ② not in dominant balance

$$\therefore \quad x \sim \log 1/\varepsilon \quad \text{as } \varepsilon \to 0^+.$$

Suggests

$$x_{n+1} = g_\varepsilon(x_n) = \log(x_n) + \log(1/\varepsilon)$$

Note $g_\varepsilon'(x_0) = 1/x$

$$\sim \frac{1}{\log(1/\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0^+$$

[but slow convergence]