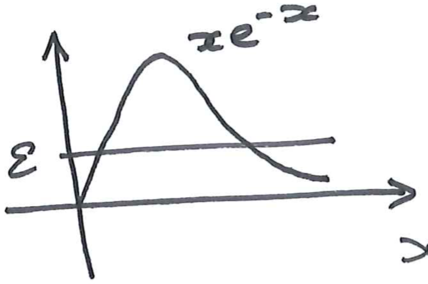
2.8 Logarithms

Example $\quad xe^{-x} = \varepsilon \qquad 0 < \varepsilon \ll 1$



Root near $x = 0$ easy to find.

$$\left[\text{Let } \underbrace{x = 0}_{\substack{\text{solution} \\ \text{if } \varepsilon = 0}} + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \right]$$

Taylor expand xe^{-x} about 0 generate powers of $\varepsilon x_1 \ldots$ which balance $\varepsilon \ldots$ higher terms integer powers … hence form of sequence

Other root $\to \infty$ as $\varepsilon \to 0$; expansion sequence not obvious

Take logs

$$\overset{①}{x} - \overset{②}{\log x} - \overset{③}{\log 1/\varepsilon} = 0$$

For x large $|①| \gg |②|$ $\therefore$ ② not in dominant balance

$$\therefore \quad x \sim \log 1/\varepsilon \quad \text{as } \varepsilon \to 0^+.$$

Suggests

$$x_{n+1} = g_\varepsilon(x_n) = \log(x_n) + \log(1/\varepsilon)$$

Note $g_\varepsilon'(x_0) = 1/x$

$$\sim \frac{1}{\log(1/\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0^+$$

[but slow convergence]

$$\therefore \quad x_0 = \log \tfrac{1}{\varepsilon}$$

$$x_1 = g_\varepsilon(x_0) = g_\varepsilon\left(\log \tfrac{1}{\varepsilon}\right) = \log\left(\tfrac{1}{\varepsilon}\right) + \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right)$$

$$x_2 = g_\varepsilon\left(\log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\right)\right)$$

$$= \log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\left(1 + \frac{\log(\log \frac{1}{\varepsilon})}{\log \frac{1}{\varepsilon}}\right)\right)$$

$\log(1+\delta) \simeq \delta$ for $|\delta| < 1$

$$= \log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\right) + \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + \cdots$$

Don't know answer... need to compute x_3 to confirm first 3 terms are correct.

Difficult sequence to guess.

Converges VERY slowly

3. Asymptotic Approximations

3.1 Definitions

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges at fixed z if $\forall \varepsilon > 0, \exists N_0(z, \varepsilon) \in \mathbb{N}$ such that $$\left| \sum_{n=m}^{N} f_n(z) \right| < \epsilon \quad \forall N \geq m > N_0$$

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges to $f(z)$ at fixed z if $\forall \varepsilon > 0, \exists N_0(z, \varepsilon) \in \mathbb{N}$ such that $$\left| f(z) - \sum_{n=0}^{N} f_n(z) \right| < \varepsilon \quad \forall N \geq N_0$$

- A series converges if its terms decay sufficiently rapidly as $n \to \infty$

- Less useful in practice than might be believed.

Example $\quad \operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \quad z \in \mathbb{C}.$

e^{-t^2} is a holomorphic function of $t \in \mathbb{C}$.

Thus it has a convergent power series with infinite radius of convergence

$$\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$$

Integrate term by term

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\, n!} = \frac{2}{\sqrt{\pi}} \left(z - z^3/3 + z^5/10 \cdots\cdots \right)$$

Has infinite radius of convergence.

For accuracy of 10^{-5}, $\quad$ 16 terms needed for $z = 2$

31 terms needed for $z = 3$

75 terms needed for $z = 5$

Cancellation required between large powers ... need lot of terms for good approximation

Alternative approach to approximating erf(z).

Rewrite $\operatorname{erf}(z) = 1 - 2/\sqrt{\pi} \int_z^\infty e^{-t^2}\,dt$

Parts

$$\int_z^\infty e^{-t^2}\,dt = \int_z^\infty \left(-\frac{1}{2t}\right)\left(-2te^{-t^2}\right)dt$$

$-1/2t \qquad -2te^{-t^2}$

$1/2t^2 \qquad e^{-t^2}$

$$= \left[-1/2t\, e^{-t^2}\right]_z^\infty - \int_z^\infty \frac{1}{2t^2} e^{-t^2}\,dt$$

$$= \frac{1}{2z} e^{-z^2} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\,dt$$

Continuing the integration by parts

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{1.3.}{(2z^2)^3} - \frac{1.3.5}{(2z^2)^5} + \cdots\right)$$

This series diverges $\forall z \in \mathbb{C}$, but truncated series very useful.

- For accuracy of 10^{-5} only two terms are needed for $z = 3$.
- Importantly The leading term is almost correct and each additional term gets us closer to the answer, with each additional correction of decreasing size until eventually they start increasing.

- This is an asymptotic series

Asymptoticness

- A sequence $\{f_n(\epsilon)\}_{n \in \mathbb{N}_0}$ is asymptotic if $\forall n \geq 1$

$$\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0$$

- A series $\sum_{n=0}^{\infty} f_n(\epsilon)$ is an <u>asymptotic expansion</u> of a function $f(\varepsilon)$ as $\varepsilon \to 0$ if $\forall N \in \mathbb{N}_0$ $\quad \dfrac{f(\varepsilon) - \sum_{n=0}^{N} f_n(\epsilon)}{f_N(\epsilon)} \to 0$ as $\varepsilon \to 0$.

- In other words, the remainder is smaller than the last term included once ε is sufficiently small.

- We write $\quad f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon)$ as $\varepsilon \to 0$

- Usually first few terms are sufficient for a good approximation

- Often $f_n(\varepsilon) = a_n \varepsilon^n$ with a_n real, in which case

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \varepsilon^n \quad \text{as } \varepsilon \to 0$$

$\left\{ \begin{array}{l} f_n = a_n \delta_n(\varepsilon) \\ \text{with } \{\delta_n(\varepsilon)\}_{n\in\mathbb{N}_0} \\ \text{asymptotic also common} \end{array} \right\}$.

is called an <u>asymptotic power series</u>.

Order Notation

- $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means
  $\exists K, \delta > 0$ s.t $|f(\varepsilon)| < K|g(\varepsilon)| \quad \forall\, |\varepsilon - \varepsilon_0| < \delta$

- $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to \varepsilon_0$

- $f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means
  $\exists K \in \mathbb{R} \setminus \{0\}$ s.t. $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to K$ as $\varepsilon \to \varepsilon_0$

Examples

$\sin(x) = O(1),\ o(1),\ O(x),\ \mathrm{ord}(x)$ as $x \to 0$

$\sin(x) = O(1)$ as $x \to \infty$

$\log(x) = o(x^{-\delta})$ as $x \to 0$ for any $\delta > 0$.

3.2 Uniqueness and manipulation of an asymptotic series

- If a function $f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ as $\varepsilon \to 0$ then induction implies that

$\{a_n\}_{n \in \mathbb{N}_0}$ is uniquely determined by $a_k = \lim_{\varepsilon \to 0} \left[\dfrac{f(\varepsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\varepsilon)}{\delta_k(\varepsilon)} \right]$

- Uniqueness is for a given sequence $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$

- The sequence need not be unique e.g.

$$\tan \varepsilon \sim \varepsilon + \varepsilon^3/3 + 2\varepsilon^5/15 + \dots \quad \text{as } \varepsilon \to 0$$

$$\tan \varepsilon \sim \sin \varepsilon + \tfrac{1}{2}(\sin \varepsilon)^3 + 3/8\,(\sin \varepsilon)^5 + \dots \quad \text{as } \varepsilon \to 0$$

- Uniqueness for a given function... two functions may share the same asymptotic expansion e.g.

$$e^{\varepsilon} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \quad \text{as } \varepsilon \to 0$$

$$e^{\varepsilon} + e^{-1/\varepsilon^2} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \quad \text{as } \varepsilon \to 0$$

- Two distinct functions with the same asymptotic power series can only differ by a function that is not holomorphic as two holomorphic functions with the same power series are identical.

- Asymptotic expansions can be naively added, subtracted, divided, multiplied and divided (though the sequence eg. the $\{\delta_n(\epsilon)\}_{n\in\mathbb{N}_0}\}$ may be larger).

- This underlies expansion method for algebraic equations.

- One series can be substituted into another, but take care with exponentials .... always expand exponents to ord(1).

<u>Example</u> $f(z) = e^{z^2}$ $\quad z = 1/\varepsilon + \varepsilon$ $\quad$ Naively $f(z) \sim e^{1/\varepsilon^2}$ at leading order $\quad \times$

$$f(z) = \exp\left(\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\right) = \exp\left(\frac{1}{\varepsilon^2}+2+\varepsilon^2\right) = e^{1/\varepsilon^2}\cdot e^2\cdot\left(1+\varepsilon^2+\frac{(\varepsilon^2)^2}{2!}+\frac{(\varepsilon^2)^3}{3!}+\dots\right)$$

- Sine and Cosine and complex exponentials require analogous care in this context.

- Asymptotic expansions can be integrated term by term with respect to ε resulting in the correct asymptotic expansion for the integral.

- In general asymptotic expansions <u>cannot</u> be differentiated safely

<u>Example</u>

$$f(\varepsilon) = \varepsilon\cos(1/\varepsilon) = O(\varepsilon) \text{ as } \varepsilon \to 0$$
$$f'(\varepsilon) = \frac{1}{\varepsilon}\sin(1/\varepsilon) + \cos(1/\varepsilon) = O(1/\varepsilon) \text{ as } \varepsilon \to 0$$

Differentiating the asymptotic expansion with the $O(\varepsilon)$ start would naively give $O(1)$ ... but the derivative is $O(1/\varepsilon)$ as $\varepsilon \to 0$.

- Terms move down an asymptotic expansion with differentiation (eg. $\frac{d}{dx}x^n = nx^{n-1}$) and thus terms at higher orders may cause problems on differentiation.

- Often, first few terms sufficient. If higher accuracy required...

  <u>optimal truncation</u> : truncate asymptotic series at smallest term

3.4 Parametric Expansions

- Integrals, differential equations and partial differential equations involve functions with one, or more, variables $f(x, \varepsilon)$ with ε a small parameter.

- There is an obvious generalisation of the definition of an asymptotic expansion by allowing the coefficients to depend on x. For fixed x

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\,\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0 \quad \text{if and only if}$$

$$\frac{1}{\delta_N(\varepsilon)}\left[f(x;\varepsilon) - \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon)\right] \to 0 \qquad \text{as } \varepsilon \to 0$$

4. Asymptotic expansions of integrals

4.1 Integration by parts

Example derivation of an asymptotic power series

f' differentiable near $\varepsilon = 0$; $f(\varepsilon) = f(0) + \int_0^\varepsilon f'(x)\,dx$

Write $1 = \frac{d}{dx}(x-\varepsilon)$

Parts $$f(\varepsilon) = f(0) + \Big[(x-\varepsilon)f'(x)\Big]_0^\varepsilon - \int_0^\varepsilon (x-\varepsilon) f''(x)\,dx$$

$$= f(0) + \varepsilon f'(0) - \int_0^\varepsilon (x-\varepsilon) f''(x)\,dx$$

Repeat

$$= \sum_{n=0}^{N} \frac{f^{(n)}(0)\,\varepsilon^n}{n!} + \frac{1}{N!}\int_0^\varepsilon (\varepsilon - x)^N f^{(N+1)}(x)\,dx$$

If remainder term exists for $\forall N \in \mathbb{N}$ and sufficiently small ε, then

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0$$

[If the series converges, it is the Taylor series about zero].

Example $$I(x) = \int_x^\infty e^{-t^4}\,dt$$

Want asymptotic series as $x \to \infty$

no taylor series !!!!

$$I(x) = \int_x^\infty \left(\frac{-1}{4t^3}\right)\left(-4t^3 e^{-t^4}\right) dt$$

$$= \left[\left(-\frac{1}{4t^3}\right)e^{-t^4}\right]_x^\infty - \int_x^\infty \left(\frac{3}{4t^4}\right)e^{-t^4}\,dt$$

$$= \frac{e^{-x^4}}{4x^3} - \frac{3}{4}\underbrace{\int_x^\infty \frac{e^{-t^4}}{t^4}\,dt} \sim \frac{e^{-x^4}}{4x^3} \text{ as } x\to\infty$$

$$\int_x^\infty \frac{e^{-t^4}}{t^4}\,dt < \frac{1}{x^4}e^{-x^4}\int_x^\infty e^{-(t^4-x^4)}\,dt$$

$$t^4-x^4 = (t-x)(t+x)(t^2+x^2)$$

Let $u = t-x$

$$\int_x^\infty e^{-(t^4-x^4)}\,dt = \int_0^\infty e^{-u(u+2x)((u+x)^2+x^2)}\,du$$

$$< \int_0^\infty e^{-u^4}\,du < \int_0^\infty e^{-u^2}\,du$$

$$\therefore \int_x^\infty \frac{e^{-t^4}}{t^4}\,dt \sim O\left(\frac{1}{x^4}e^{-x^4}\right) \ll \frac{e^{-x^4}}{4x^3}$$

Correction much smaller than "last" term

Further integration by parts will give higher order terms.

Example $$I(x) = \int_0^x t^{-1/2} e^{-t}\,dt$$

Naive Integration by parts fails.

$$I(x) = \left[-t^{-1/2} e^{-t}\right]_0^x - \int_0^x \underbrace{\left(-\tfrac{1}{2} t^{-3/2}\right)}_{\text{Not integrable}} \left(-e^{-t}\right) dt$$

divergence at $x=0$ not really an issue ... take care of it separately

$$\therefore I(x) = \underbrace{\int_0^\infty t^{-1/2} e^{-t}\,dt}_{\Gamma(1/2) = \sqrt{\pi}} - \underbrace{\int_x^\infty t^{-1/2} e^{-t}\,dt}_{J(x)}$$

use substitution $u = \sqrt{t}$

$$J(x) = \int_x^\infty t^{-1/2} e^{-t}\,dt = \left[-t^{-1/2} e^{-t}\right]_x^\infty - \frac{1}{2}\int_x^\infty t^{-3/2} e^{-t}\,dt$$

$$= \frac{e^{-x}}{\sqrt{x}} - \frac{1}{2}\underbrace{\int_x^\infty \frac{e^{-t}}{t^{3/2}}\,dt}$$

$$< \frac{1}{x^{3/2}}\int_x^\infty e^{-t}\,dt = \frac{e^{-x}}{x^{3/2}} \ll \frac{e^{-x}}{x^{1/2}}$$

$$\frac{\text{Correction}}{\text{Last term}} \to 0 \quad \text{as } x \to \infty$$

$$\therefore I(x) \sim \sqrt{\pi} - e^{-x}/\sqrt{x} + \dots$$

<u>General Rule</u> Integration by parts works if the contribution from one of the limits of the integration dominates

4.2 Failure of Integration by Parts

Example $\quad I(x) = \int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ for $x > 0$.

let $u = x^{1/2} t$

Attempt (Parts)

$$I(x) = \int_0^\infty \left(\frac{-1}{2xt}\right)\left(-2xte^{-xt^2}\right)dt$$

$$= \underbrace{\left[\frac{e^{-xt^2}}{-2xt}\right]_0^\infty - \int_0^\infty \frac{e^{-xt^2}}{2xt^2}\,dt}$$

does not exist; fractional power in x not picked up by this type of expansion

$\therefore$ Integration by parts simple but inflexible, of limited use.

Also does not work when dominant contribution to integral is from domain interior (need one limit to dominate).

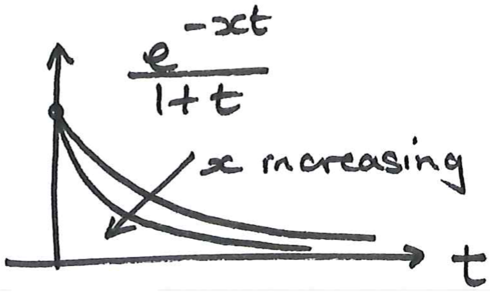
4.3 Laplace's Method

General technique for the asymptotic expansion as $x \to \infty$ of

$$I(x) = \int_a^b f(t)\,e^{x\varphi(t)}\,dt$$

with $[a,b] \subseteq \mathbb{R}$ and f, φ continuous real functions on $[a,b]$.

Example $\quad I(x) = \int_0^1 \frac{e^{-xt}}{1+t}\,dt$



main contribution

$$I(x) = \overbrace{\int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\,dt}^{I_1(x)} + \overbrace{\int_{\varepsilon}^{1} \frac{e^{-xt}}{1+t}\,dt}^{I_2(x)}$$

with $0 < 1/x \ll \varepsilon \ll 1.$

$$I_1(x) = \frac{1}{x}\int_0^{x\varepsilon} \frac{e^{-s}}{1+s/x}\,ds$$

$s/x \leq x\varepsilon/x = \varepsilon \ll 1$

$\therefore$ within radius of convergence and expansion uniform.

$$= \frac{1}{x}\int_0^{x\varepsilon} e^{-s}\left(\sum_{n=0}^{\infty}\left(\frac{-s}{x}\right)^n\right) ds$$

$$= \frac{1}{x}\sum_{n=0}^{\infty}\underbrace{\left[\int_0^{x\varepsilon} s^n e^{-s}\,ds\right]}_{\underbrace{\int_0^{\infty} s^n e^{-s}\,ds}_{n!} - \underbrace{\int_{x\varepsilon}^{\infty} s^n e^{-s}\,ds}_{\leq (x\varepsilon)^n e^{-\varepsilon x} + \int_{x\varepsilon}^{\infty} s^{n-1} e^{-s}\,ds \;\ll\; n!}} \frac{(-1)^n}{x^n}$$

For fixed n, making x larger will ensure this is true... exponentially small... will not compete with powers from further down the series.

$$\therefore\; I_1(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} \quad \text{as } x \to \infty$$

$$I_2(x) < \int_{\varepsilon}^{1} e^{-xt}\,dt = e^{-x\varepsilon} - e^{-x}$$

already dropped terms larger than this

$\ll I_1(x)$

This is smaller still.

$$\therefore\; I(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} \quad \text{as } x \to \infty$$