4.4 Watson's Lemma

Let $\quad I(x) = \int_0^b f(t) e^{-xt}\, dt \quad , \quad b > 0,$

with

(i) $f(t)$ continuous on $t \in [0, b]$

(ii) If $b = \infty$, in addition $\exists c \in \mathbb{R}$ with $f(t) = o(e^{ct})$ as $t \to \infty$

(iii)
$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as } t \to 0^+$$

with $\alpha > -1$, $\beta > 0$, $a_n \in \mathbb{R}$ for $n \in \mathbb{N}_0$.

Then
$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}} \quad \text{as } x \to \infty$$

where $\quad \Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t}\, dt.$

Note $\quad \Gamma(m) = (m-1)!$ for $m \in \mathbb{N}$.

Proof See Supplementary Notes online.

If f uniformly convergent in neighbourhood of origin, proceeds as in example above

4.5 General Laplace Integrals

- Dominant contribution to

$$I(x) = \int_a^b f(t) e^{x\varphi(t)}\, dt \quad \text{as } x \to \infty$$

is from the region where $\varphi(t)$ is the largest.

- There are 3 cases: the maximum of $\varphi(t)$ is at

(i) $t = a$, (ii) $t = b$, (iii) $t = c \in (a,b)$.

- To proceed
  - dominant contribution from near maximum of φ
  - reduce range of integration to this region only gives exponentially small errors
  - within this region, Taylor expand φ, f.
  - After rescaling replace limits by $\pm\infty$

need to assess size of ε relative to $1/x$ --- later!

Case (i) with $\varphi'(a) < 0$, $f(a) \neq 0$, $\varphi''(a) \neq 0$

$$I(x) = \underbrace{\int_a^{a+\varepsilon} f(t) e^{x\varphi(t)}\, dt}_{I_1(x)} + \underbrace{\int_{a+\varepsilon}^b f(t) e^{x\varphi(t)}\, dt}_{I_2(x)}$$

$|I_1| \gg |I_2|$

$$e^{x\varphi(a+\varepsilon)} \ll e^{x\varphi(a)} \qquad \varphi(a+\varepsilon) \simeq \varphi(a) + \varepsilon\varphi'(a)$$

$$e^{x\varepsilon\varphi'(a)} \ll 1$$

$$\boxed{x\varepsilon \gg 1}$$

$$I_1(x) = \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \dots] \exp\left[x\left\{\varphi(a) + (t-a)\varphi'(a) + \frac{(t-a)^2}{2}\varphi''(a) + \dots\right\}\right] dt$$

$$= e^{x\varphi(a)} \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \dots] e^{x(t-a)\varphi'(a)} \left[1 + x\frac{(t-a)^2}{2}\varphi''(a) + \dots\right] dt$$

okay given $\boxed{x\varepsilon^2 \ll 1}$

$\therefore$ $\boxed{\frac{1}{x} \ll \varepsilon \ll \frac{1}{\sqrt{x}}}$

Rescale $\boxed{x(t-a) = s}$ ← Remove x from leading exponent.

$$= \frac{e^{x\varphi(a)}}{x} \int_0^{\varepsilon x} [f(a) + O(s/x)] e^{s\varphi'(a)} \left[1 + O\left(\frac{s^2}{x}\right)\right] ds$$

$\to\infty$ OKAY as $\varepsilon x \gg 1$

$$= \frac{f(a)e^{x\varphi(a)}}{x} \left(\int_0^{\varepsilon x} e^{s\varphi'(a)} \left(1 + O\left(\frac{1}{x}\right)\right) ds \right)$$

$$= \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \left(1 + O\left(\frac{1}{x}\right)\right)$$

← guarantees asymptoticness... correction much smaller than last term.

$$\therefore \quad I(x) \sim I_1(x) \sim \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \quad \text{as } x \to \infty.$$

Case (ii) with $\varphi'(b) > 0$, $f(b) \neq 0$, $\varphi''(b) \neq 0$. **Exercise** Show that

$$I(x) \sim \frac{f(b)e^{x\varphi(b)}}{x\,\varphi'(b)} \quad \text{as } x \to \infty.$$

← Essentially identical to case (i)

Case (iii) $\varphi'(c) = 0, \quad \varphi''(c) < 0, \quad f(c) \neq 0, \quad \varphi'''(c) \neq 0$

$t = c$ global maximum of $\varphi(t)$ for $t \in [a,b]$.

$$I(x) = \Bigg[\underbrace{\int_a^{c-\varepsilon} dt}_{I_1} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon} dt}_{I_2} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3} \Bigg] f(t) e^{x\varphi(t)}$$

I_2 dominant

$$e^{x\varphi(c+\varepsilon)} \ll e^{x\varphi(c)} \quad \text{for } |I_2| \gg |I_3|$$

$\varphi(c+\varepsilon) \approx \varphi(c) + \frac{\varepsilon^2}{2}\varphi''(c)$ as $\varphi'(c) = 0$

$$\therefore \quad e^{\frac{x\varepsilon^2\varphi''(c)}{2}} \ll 1 \quad \therefore \quad \boxed{x\varepsilon^2 \gg 1}$$

Same argument for $|I_2| \gg |I_1|$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt\, f(t) e^{x\varphi(t)}$$

$$= \int_{c-\varepsilon}^{c+\varepsilon} \left[f(c) + O(t-c) \right] e^{x\varphi(c)}\, e^{x(t-c)^2/2\, \varphi''(c)} \cdot \Big[1 + \underbrace{O\big(x(t-c)^3/3!\big)}_{x\varepsilon^3 \ll 1} \Big] dt$$

$$\therefore \quad \underbrace{\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll \frac{1}{x^{1/3}}}}_{\text{Need } x \text{ rather large}}$$

eg suppose $x = 8$

$\frac{1}{2\sqrt{2}} \ll \varepsilon \ll 1/2$

but $\frac{1}{\sqrt{2}} \not\ll 1$

Rescale $s = \sqrt{x}(t-c)$

$$I_2(x) = \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\varepsilon}^{\sqrt{x}\varepsilon} ds \; e^{s^2/2\,\varphi''(c)} \underbrace{\left(1 + O\left(\frac{s}{\sqrt{x}}\right)\right)}_{\text{from expansion of } f} \left(1+O\left(\frac{s^3}{\sqrt{x}}\right)\right)$$

from exponential expansion

$$= \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \underbrace{\int_{-\infty}^{\infty} ds\, e^{s^2/2\,\varphi''(c)}}_{\sqrt{\frac{2}{-\varphi''(c)}} \int_{-\infty}^{\infty} du\, e^{-u^2}} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

okay as $\sqrt{x}\varepsilon \gg 1$

Substitute $-s^2/2\,\varphi''(c) = u^2$

$$= \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

$$\therefore I(x) \sim I_2(x) \sim \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \quad \text{as } x \to \infty$$

4.6 Method of Stationary Phase

- Used when $\varphi = i\psi$, ψ real, so that

$$I(x) = \int_a^b f(t)\, e^{ix\psi(t)}\, dt.$$

Riemann-Lebesgue Lemma

If $\int_a^b |f(t)|\, dt < \infty$ and $\psi(t)$ is continuously differentiable for $t \in [a,b]$ and not constant on any sub-interval of $[a,b]$ then

$$\int_a^b f(t)\, e^{ix\psi(t)}\, dt \to 0 \quad \text{as } x \to \infty.$$

Useful

- Useful for integration by parts, eg.

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\, dt = \underline{-\frac{ie^{ix}}{2ix} + \frac{i}{x}} - \frac{i}{x} \underbrace{\int_0^1 \frac{e^{ixt}}{(1+t)^2}\, dt}_{\to 0 \text{ as } x \to \infty \text{ by RLL.}}$$

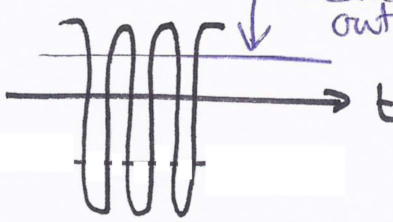
First term of an asymptotic expansion

- Why does RLL hold?

(i) For $\psi(t) = t$.

$$\int_a^b f(t)\, \underbrace{e^{ixt}}_{\text{oscillates more and more rapidly}}\, dt$$

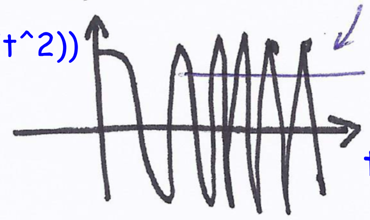
$f(t) \doteq$ const. over period $\therefore$ Cancels out.



(ii) More generally.

Near $t = t_0$, $\psi(t) \sim \psi(t_0) + (t-t_0)\psi'(t_0) + \cdots$

Period of oscillation $\sim \dfrac{2\pi}{x|\psi'(t_0)|}$



$\to 0$ as $x \to \infty$

providing $|\psi'(t_0)| \neq 0$

$\therefore$ Again get cancellation, unless $|\psi'(t_0)| = 0$

Unless ψ is constant on a region of non-zero measure, a stationary point is not enough to save the integral as $x \to \infty$, and one gets zero.

Nonetheless the dominant terms for x large but not infinite are from where $|\psi'(t_0)| = 0$

Example

$\psi''(t) \sim O(1)$ in neighbourhood of c.

$f(c) \neq 0$; $\psi'(c) = 0$, $c \in (a,b)$; $\psi'(t) \neq 0$ $t \in [a,b] \backslash \{c\}$.

$$I(x) = \Big[\underbrace{\int_a^{c-\varepsilon}}_{I_1(x)} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon}}_{I_2(x)} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3(x)} \Big] f(t) e^{ix\psi(t)} dt$$

$\varepsilon \ll 1$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c) + O(t-c)\right] \exp\left[ix\left\{\psi(c) + \tfrac{1}{2}(t-c)^2\psi''(c) + O\left((t-c)^3\right)\right\}\right].$$

1) Reduce range of integration (Need to check errors later)

2) Taylor Expand

$$= e^{ix\psi(c)} \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c)+O(t-c)\right] e^{\frac{ix}{2}(t-c)^2\psi''(c)} \left(1+O\left(\underbrace{(t-c)^3 x}\right)\right)$$

providing $\varepsilon^3 x \ll 1$

$\therefore \varepsilon \ll 1/x^{1/3}$

$x(t-c)^2 = s^2$

3) Rescale

$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} ds \left(f(c) + \underbrace{O\left(\frac{s}{\sqrt{x}}\right)}_{O(\varepsilon)}\right) e^{is^2\psi''(c)/2} \left(1 + \underbrace{O\left(s^3/\sqrt{x}\right)}_{O(\varepsilon^3 x)\ll 1}\right)$$

Drop

4) Replace Limits

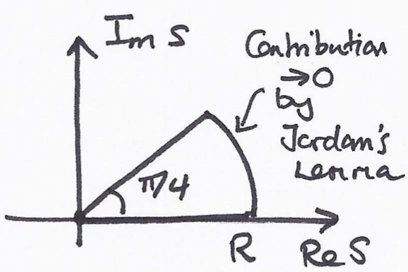
$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c) \int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} + \dots$$

$\varepsilon\sqrt{x} \gg 1$

$$\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}}$$

$$\int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} = 2\int_0^{\infty} ds\, e^{is^2\psi''(c)/2}$$

$$= \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} e^{i\pi/4\,\mathrm{sgn}(\psi''(c))}$$



$\psi''(c) > 0$

Angle $-\pi/4$ for $\psi''(c) < 0$

$$\therefore \quad I_2(x) = \frac{2\pi}{|\psi''(c)|^{1/2}} \exp\left[i\pi/4 \,\mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi''(c)}}{\sqrt{x}} f(c) + \ldots$$

Size of Correction terms

1) Corrections from change of limits

$$\int_{\varepsilon\sqrt{x}}^{\infty} e^{is^2\psi''(c)/2}\, ds = \int_{\varepsilon\sqrt{x}}^{\infty} \frac{ds}{is\psi''(c)}\, is\psi''(c) e^{is^2\psi''(c)/2}$$

$$= \left[\frac{1}{is\psi''(c)} e^{is^2\psi''(c)/2}\right]_{\varepsilon\sqrt{x}}^{\infty} - \underbrace{\int_{\varepsilon\sqrt{x}}^{\infty} \frac{-1}{is^2\psi''(c)} e^{is^2\psi''(c)/2}\, ds}_{\text{Smaller correction}}$$

$$= O\left(\frac{1}{\varepsilon\sqrt{x}}\right) \sim$$

(*) for $n=0$

Similar contribution from $\displaystyle\int_{-\infty}^{-\sqrt{x}\varepsilon} e^{is^2\psi''(c)/2}\, ds$

2) Corrections from Taylor Expansions

Expansion in $(t-c)^3 x = s^3/\sqrt{x}$

$$\underbrace{\frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \frac{s^n}{x^{n/2}} e^{is^2\psi''(c)/2}\, ds}, \qquad \underbrace{\frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \frac{(s^3)^n}{x^{n/2}} e^{is^2\psi''(c)/2}}$$

$$\frac{1}{\sqrt{x}} \frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{n-1} \sim \frac{\varepsilon^{n-1}}{x} \qquad \frac{1}{\sqrt{x}} \frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{3(n-1)} \sim \frac{1}{x} (x\varepsilon^3)^{n-1}$$

Using $\displaystyle\int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} s^n e^{is^2\psi''(c)/2}\, ds \overset{(*)}{=} O\left((\sqrt{x}\varepsilon)^{n-1}\right)$ by parts.

3) Correction from $I_1(x)$

$$I_1(x) = \int_a^{c-\varepsilon} f(t)\, e^{ix\psi(t)}\, dt \qquad \frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}$$

$$= \int_a^{c-\varepsilon} \frac{f(t)}{ix\psi'(t)} \frac{\partial}{\partial t}\left(e^{ix\psi(t)}\right) dt$$

$$= \left[\frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\right]_a^{c-\varepsilon} - \frac{1}{ix}\underbrace{\underbrace{\int_a^{c-\varepsilon} e^{ix\psi(t)} \frac{\partial}{\partial t}\left(\frac{f(t)}{\psi'(t)}\right) dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL if it exists}}}_{o\left(\frac{1}{x}\right)}$$

small "oh"

$$\sim O\left(\frac{1}{x\psi'(c-\varepsilon)}\right)$$

$$\sim O\left(\frac{1}{\varepsilon x}\right)$$

$\psi'' \sim O(1)$ in neighbourhood of c.

Similarly for I_3.

$\therefore$ Corrections $\sim O\left(\frac{1}{\varepsilon x}, \frac{1}{\varepsilon\sqrt{x}}, \frac{1}{x}\right) \sim O\left(\frac{1}{\varepsilon\sqrt{x}}\right)$

Note Corrections algebraically small, not exponentially small as in other methods

Next order terms very difficult to find, in general need to consider whole integration domain not just behaviour near $t=c$

$$\boxed{\therefore\ I(x) \sim \frac{2\pi}{|\psi''(c)|^{1/2}} e^{i\pi/4\,\mathrm{sgn}(\psi''(c))} \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c) \quad \text{with corrections at } O\left(\frac{1}{\varepsilon\sqrt{x}}\right)}$$