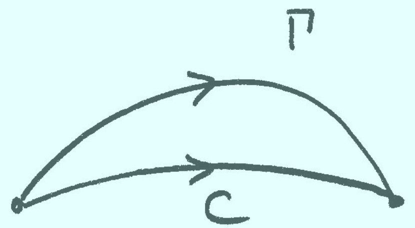
4.7 Method of Steepest Descents

- Generalises Laplace's method to consider

$$I(x) = \int_C f(t)\, e^{x\varphi(t)}\, dt \qquad \text{as } x \to \infty,\ x \text{ real},$$

where $f(t), \varphi(t)$ are holomorphic (and thus analytic), with C a contour in the complex t plane.

- Key idea $I(x)$ unchanged upon deforming C to a new contour Γ, with the same start and end points.



$$I(x) = \int_\Gamma f(t)\, e^{x\varphi(t)}\, dt$$

- If we find a contour Γ on which $\text{Im}(\varphi(t))$ is piecewise constant, i.e. Γ_j, v_j such that $\Gamma = \bigcup_j \Gamma_j$ with $\text{Im}\,\varphi(t) = v_j = \text{const}$ on Γ_j then

$$I(x) = \sum_j e^{ixv_j} \int_{\Gamma_j} f(t) e^{x \operatorname{Re} \varphi(t)} dt$$

and each integral can be analysed as $x \to \infty$ using Laplace's method.

- Let $\varphi(t) = u(\xi, \eta) + iv(\xi, \eta)$ with $t = \xi + i\eta$.
- As φ is holomorphic, we have the Cauchy Riemann Equations (CRE):

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}.$$

Hence $\nabla u \cdot \nabla v = u_\xi v_\xi + u_\eta v_\eta = 0$ $\therefore$ $\nabla u \perp \nabla v$

Also $\nabla v \perp$ contours with v const $\therefore$ Contours with v const $\parallel \nabla u$.

∇u points in direction u increases at fastest rate

$-\nabla u$ points in direction u decreases at fastest rate

$\therefore$ Contour with v constant is a path of steepest ascent/descent of u.

Landscape of $u(\xi,\eta)$

- CRE. $\quad u_{\xi\xi} + u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0$

- Hence u cannot have a maximum or a minimum (unless we are also considering a point where u is singular or a branch point, where u is not holomorphic).

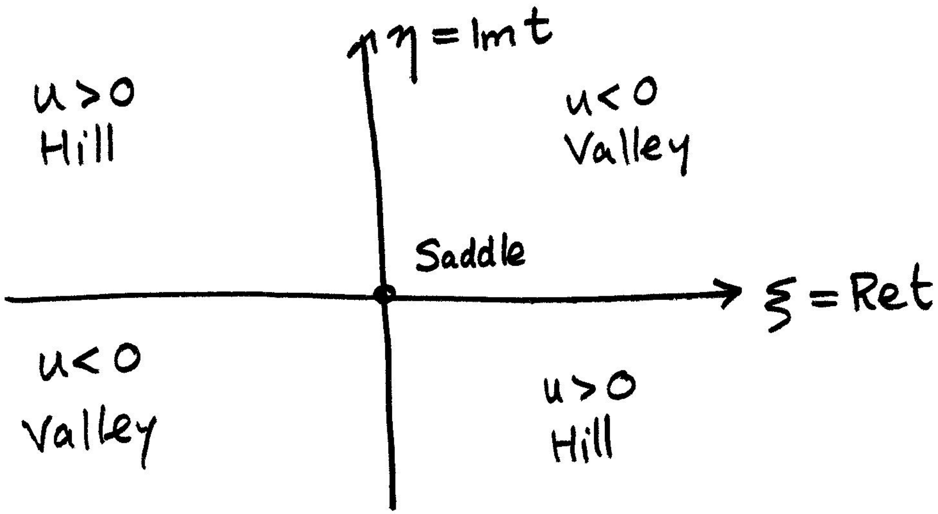
- At a stationary point, where $u_\xi = u_\eta = 0$, we have a SADDLE.

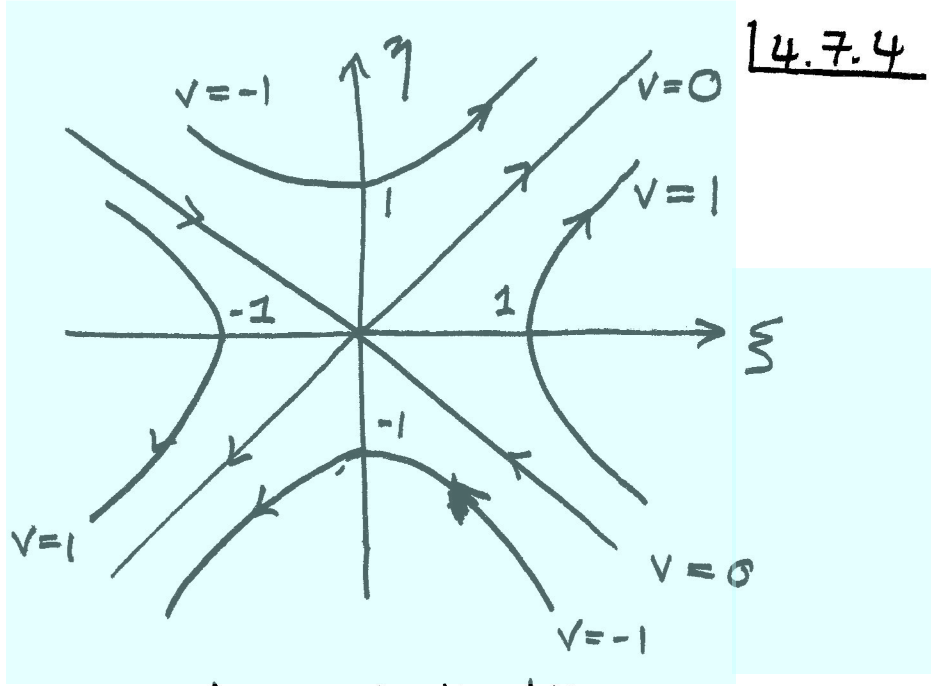
- Landscape of u has hills $(u>0)$, valleys $(u<0)$ at infinity with saddle points in the interior of the complex plane.

Example

$$\varphi(t) = it^2 = i(\xi+i\eta)^2 = -2\xi\eta + i(\xi^2-\eta^2) \quad \therefore u = -2\xi\eta,\ v = \xi^2-\eta^2$$

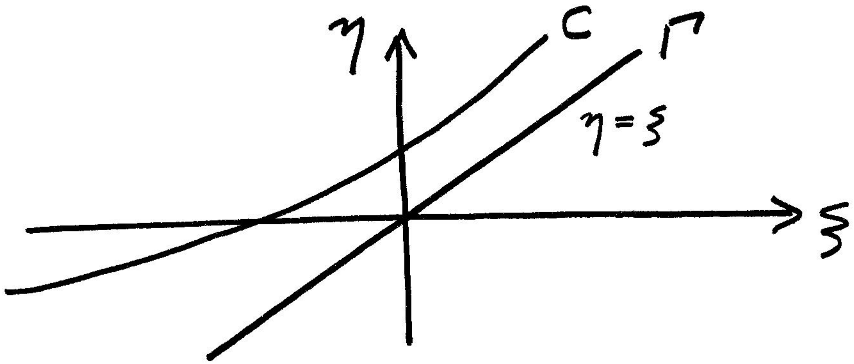
$$\nabla u = -2(\eta,\xi) \quad \therefore \text{Saddle point at } \xi=\eta=0$$





Arrows in direction of decreasing u with STEEPEST DESCENT

- Contour C infinite, with endpoints in different valleys.
  - If endpoints not in valleys, integral $I(x)$ not well defined.



Deform C into Γ.

Integrals at infinity subleading

Hence method known as "Method of steepest descents" or saddle point method

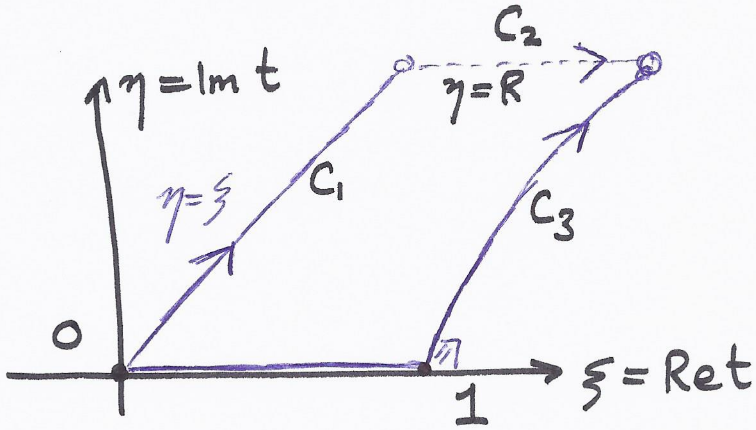
To use the method ...

* Deform contour to union of steepest descent (v const) contours through the endpoints and any relevant saddle points

* Evaluate local contributions from saddle and end points using Laplace's method.

Example

$$I(x) = \int_0^1 e^{x\varphi(t)}\,dt \quad \text{as } x\to\infty, \text{ with } \varphi(t) = it^2.$$



Steepest descent contour through $t=0$ is $\eta = \xi$

Steepest descent contour through $t=1$ is $\xi^2 - \eta^2 = 1$

$$C_1(R) = \{\, \xi(1+i),\ \ \xi \in [0,R] \,\}$$

$$C_2(R) = \{\, \xi + iR,\ \ \xi \in [R, \sqrt{R^2+1}\,] \,\}$$

$$C_3(R) = \{\, \sqrt{1+\eta^2} + i\eta,\ \ \eta \in [0,R] \,\}$$

$$\therefore\ I(x) = \left[\int_{C_1(R)} + \int_{C_2(R)} - \int_{C_3(R)} \right] e^{ixt^2}\,dt$$

On $C_2(R)$ $\quad |\exp(ixt^2)| = |\exp(ix(\xi^2 - R^2 + 2i\xi R))|$

$$= |\exp(-2x\xi R)| = O\!\left(e^{-2xR^2}\right) \to 0$$

as $R\to\infty$.

$$\therefore\ \int_{C_2(R)} e^{ixt^2}\,dt \to 0 \quad \text{as } R\to\infty.$$

$$\int_{C_1(\infty)} e^{ixt^2}dt = \int_0^\infty \exp\left(ix\xi^2(1+i)^2\right) d\xi\,(1+i)$$

$i(1+i)^2 = i(1+2i+i^2)$
$= 2i^2 = -2.$

$$= (1+i)\int_0^\infty e^{-2x\xi^2} d\xi \qquad u = \sqrt{2x}\,\xi$$

$$= \frac{1+i}{\sqrt{2}\sqrt{x}} \int_0^\infty e^{-u^2} du = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$$\int_{C_3(\infty)} e^{ixt^2}dt = \int_0^\infty e^{ix\underbrace{\left[(1+\eta^2)^{1/2} + i\eta\right]^2}_{1+2i\eta(1+\eta^2)^{1/2}}} \frac{dt}{d\eta}\, d\eta$$

$$= e^{ix}\int_0^\infty e^{x\varphi(\eta)} f(\eta)\, d\eta$$

with $\varphi(\eta) = -2\eta(1+\eta^2)^{1/2}$,

$$f(\eta) = \frac{dt}{d\eta} = \frac{\eta}{(1+\eta^2)^{1/2}} + i$$

and thus Laplace's method can be used.

However, we can get to a quicker answer, at all orders, by noting

on $C_3(\infty)$, $\qquad t = \xi + i\eta \qquad$ where $\xi^2 - \eta^2 = 1$

$$\therefore\ t^2 = \xi^2 - \eta^2 + 2i\xi\eta = 1 + 2i\eta(1+\eta^2)^{1/2}$$

$\therefore$ Let $t^2 = 1 + is \qquad s \in [0,\infty)$

$\therefore$ $\underline{\underline{t = (1+is)^{1/2}}}$ (principal branch ie +ve squareroot).

$$\frac{dt}{ds} = \tfrac{1}{2}\, i \,\frac{1}{(1+is)^{1/2}}$$

Then

$$\int_{C_3(\infty)} e^{ixt^2}\,dt = \int_0^\infty e^{ix}\cdot e^{-xs}\,\frac{dt}{ds}\,ds$$

$$= \frac{ie^{ix}}{2}\int_0^\infty e^{-xs}\,\frac{1}{(1+is)^{1/2}}\,ds$$

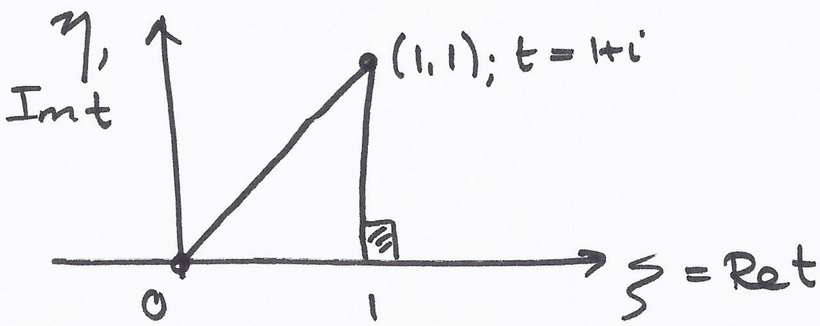
$$\underset{\text{Lemma}}{\overset{\text{Watson's}}{\sim}} \frac{ie^{ix}}{2}\sum_{n=0}^{\infty}\frac{a_n\Gamma(n+1)}{x^{n+1}} \quad \text{as } x\to\infty$$

$$\text{with } a_n = \frac{(-i)^n\,\Gamma(n+\tfrac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}$$

$$\therefore I(x) \sim \frac{e^{i\pi/4}}{2}\sqrt{\pi/x} - \frac{ie^{ix}}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-i)^n\,\Gamma(n+\tfrac{1}{2})}{x^{n+1}} \quad \text{as } x\to\infty$$

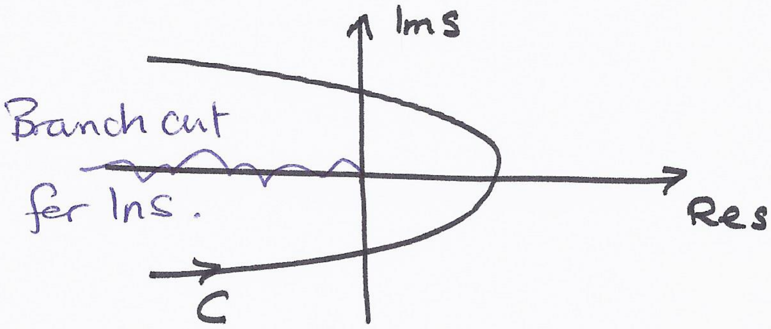
<u>Note</u>

Local contributions dominate ... just need to get tangents to steepest descent paths ... eg. could use



in the above example.

Example



$$I(x) = \int_C e^s s^{-x}\, ds$$

as $x \to \infty$

Note $\quad e^s s^{-x} = \exp\left[s - x \log s\right]$, branch cut for $\log s$, is given by $\{\mathrm{Re}\, s < 0, \mathrm{Im}\, s = 0\}$

Saddle point at $x/s = 1$. Fix saddle point location by setting $t = s/x$.

Let $s = tx$

$$I(x) = x \int_{C_*} dt\, \underbrace{e^{tx - x\log(tx)}}_{e^{tx - x\log t - x\log x}} = x^{1-x} \int_{C_*} dt\, e^{x\varphi(t)}$$

with $\varphi(t) = t - \log t$.

$\therefore \varphi = \xi + i\eta - \log r - i\theta$

polars.

$0 = \varphi'(t) = 1 - 1/t \quad \therefore$ Saddle at $t = 1$

Deform C_* through this point

$$u = \mathrm{Re}\,\varphi = r\cos\theta - \log r \qquad v = \mathrm{Im}\,\varphi = r\sin\theta - \theta$$

At $t = 1$ $\quad \theta = 0, \; v = 0$

$\therefore$ Path of steepest descent through $t = 1$ given by

$$r = \frac{\theta}{\sin\theta} \qquad \theta \in (-\pi, \pi)$$

On this path, Γ

$$u = \mathrm{Re}\,\varphi = r(\theta)\cos\theta - \log r(\theta)$$
$$= \theta\cot\theta - \log\theta + \log\sin\theta$$

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$$\therefore \quad I(x) = x^{1-x} \int_{-\pi}^{\pi} e^{x u(\theta)} \frac{dt}{d\theta} \, d\theta$$

$t = r(\theta) e^{i\theta}$

$dt/d\theta = (r'(\theta) + i r(\theta)) e^{i\theta}$

$$= x^{1-x} \int_{-\pi}^{\pi} d\theta \; e^{x \{ \underbrace{\theta \cot\theta - \log\left(\frac{\theta}{\sin\theta}\right)}_{\Phi(\theta)} \}} \underbrace{\left[r'(\theta) + i r(\theta) \right] e^{i\theta}}_{F(\theta)}$$

Laplace's method with interior maximum at $\theta = 0$

$$I(x) \sim x^{1-x} \frac{\sqrt{2\pi}\, F(0) e^{x \Phi(0)}}{\sqrt{-\Phi''(0) x}} \qquad \text{as } x \to \infty$$

By Taylor expanding, $r(\theta) = \dfrac{\theta}{\sin\theta} = \dfrac{\theta}{\theta - \theta^3/3! + \cdots} = 1 + \theta^2/6 + O(\theta^3)$

and hence $F(0) = i$

$$\Phi(\theta) = \frac{\theta(1 - \theta^2/2! + \cdots)}{\theta - \theta^3/3! + \cdots} - \log\left(1 + \theta^2/6 + O(\theta^3)\right)$$

$$= 1 - \theta^2/2 + O(\theta^3)$$

$\Phi(0) = 1 \qquad \Phi''(0) = -1$

$$\therefore \quad I(x) \sim i x^{1/2 - x} e^{x} \sqrt{2\pi} \qquad \text{as } x \to \infty$$

NB this example can be used to deduce $\Gamma(x) \sim \sqrt{2\pi}\, x^{x-1/2} e^{-x}$, ie. Stirling's approx... see online notes.

4.8 Splitting Integration Range

* Previously, have split integration range to isolate dominant contribution
* More generally, can split integration range and use different approximations in each range

$x \sim O(1)$ Integrand $O(1)$
Integration range $O(1)$
Integral $O(1)$

Example $$I(\varepsilon) = \int_0^1 \frac{dx}{(x+\varepsilon)^{1/2}} \quad \text{as } \varepsilon \to 0^+.$$

$x \sim O(\varepsilon)$ Integrand $O(1/\varepsilon^{1/2})$

$x = \text{ord}(1)$
$$\frac{1}{(x+\varepsilon)^{1/2}} = \frac{1}{x^{1/2}} \frac{1}{(1+\varepsilon/x)^{1/2}}$$
$$= \frac{1}{x^{1/2}} \left(1 - \frac{\varepsilon}{2x} + O(\varepsilon^2/x^2)\right)$$
as $\varepsilon \to 0$

Expansion not valid for $x \sim O(\varepsilon)$

$\therefore$ Split.
$$I(x) = \underbrace{\int_0^\delta \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_1} + \underbrace{\int_\delta^1 \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_2} \qquad \varepsilon \ll \delta \ll 1$$

$O(1)$

$$I_2 = \int_\delta^1 dx \left(\frac{1}{x^{1/2}} - \frac{\varepsilon}{2x^{3/2}} + O(\varepsilon^2/x^{5/2})\right)$$
okay as $\frac{\varepsilon}{x} < \varepsilon/\delta \ll 1$

$$= 2(1-\delta^{1/2}) + \varepsilon\left(1 - \frac{1}{\sqrt{\delta}}\right) + O(\varepsilon^2/\delta^{3/2})$$

$$I_1 = \int_0^{\delta} \frac{dx}{(x+\varepsilon)^{1/2}}$$ Let $x = \varepsilon u$

$$= \int_0^{\delta/\varepsilon} \frac{\varepsilon\, du}{\varepsilon^{1/2}(1+u)^{1/2}} = 2\varepsilon^{1/2}(1+\delta/\varepsilon)^{1/2} - 2\varepsilon^{1/2}$$

$\varepsilon/\delta \ll 1$

$$= 2\delta^{1/2}(1+\varepsilon/\delta)^{1/2} - 2\varepsilon^{1/2}$$

$$= 2\delta^{1/2} + \varepsilon/\delta^{1/2} + O(\varepsilon^2/\delta^{3/2}) - 2\varepsilon^{1/2}$$

$$\therefore I = I_1 + I_2 = 2 - 2\delta^{1/2} + \varepsilon - \varepsilon/\delta^{1/2} + O(\varepsilon^2/\delta^{3/2})$$
$$+ 2\delta^{1/2} + \varepsilon/\delta^{1/2} - 2\varepsilon^{1/2} + O(\varepsilon^2/\delta^{3/2})$$

$$= 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots$$ noting $\varepsilon \ll \delta$

$$\therefore \frac{\varepsilon^2}{\delta^{3/2}} = \frac{\varepsilon^2}{\delta^2}\delta^{1/2} \ll 1$$

NB Exact value

$$I(\varepsilon) = 2((1+\varepsilon)^{1/2} - \varepsilon^{1/2}) = 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots$$

Example

$$I(\varepsilon) = \int_0^{\pi/4} \frac{d\theta}{\varepsilon^2 + \sin^2\theta}$$ as $\varepsilon \to 0$

$\theta \sim O(1)$ Integrand $\sim O(1)$ Integral $\sim O(1)$

$\theta \sim O(\varepsilon)$ Integrand $\sim O(\frac{1}{\varepsilon^2})$ Integration range $\sim O(\varepsilon)$

Integral $\sim O(1/\varepsilon)$

Split

$$I = \underbrace{\int_0^\delta \frac{d\theta}{\varepsilon^2 + \sin^2\theta}}_{I_1} + \underbrace{\int_\delta^{\pi/4} \frac{d\theta}{\varepsilon^2 + \sin^2\theta}}_{I_2}$$

$\varepsilon \ll \delta \ll 1$

need this to be small ... thus need

$$I_2 = \int_\delta^{\pi/4} \left(\frac{1}{\sin^2\theta} + O\left(\frac{\varepsilon^2}{\sin^4\theta}\right)\right) d\theta$$

$\varepsilon^2/\delta^4 \ll 1$

i.e. $\varepsilon^{1/2} \ll \delta \ll 1$

$$= -\left[\cot\theta\right]_\delta^{\pi/4} + O(\varepsilon^2/\delta^3)$$

$$= -1 + \frac{(1 - \delta^2/2 + \dots)}{\delta - \delta^3/6 + \dots} + O(\varepsilon^2/\delta^3) = -1 + \frac{1}{\delta} + O(\delta) + \underbrace{O(\varepsilon^2/\delta^3)}$$

$\ll 1$ for $\varepsilon^{2/3} \ll \delta \ll 1$

consistent with $\varepsilon^{1/2} \ll \delta \ll 1$

wlog $\varepsilon > 0$ $\theta = \varepsilon u$

$\varepsilon u < \varepsilon \cdot \frac{\delta}{\varepsilon} = \delta \ll 1$

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon \, du}{\varepsilon^2 + \sin^2(\varepsilon u)}$$

$$= \varepsilon \int_0^{\delta/\varepsilon} \frac{du}{\varepsilon^2 + \varepsilon^2 u^2 + O(\varepsilon^4 u^4)}$$