$$I_1 = \frac{1}{\varepsilon}\int_0^{\delta/\varepsilon} du \left[\frac{1}{1+u^2} + O\left(\frac{\varepsilon^2 u^4}{(1+u^2)^2}\right)\right]$$

$$= \frac{1}{\varepsilon}\tan^{-1}(\delta/\varepsilon) + O\left(\frac{1}{\varepsilon}\cdot\frac{\delta}{\varepsilon}\cdot\varepsilon^2\right)$$

$$= \frac{1}{\varepsilon}\left[\frac{\pi}{2} - \frac{1}{\delta/\varepsilon} + O\left(\frac{1}{(\delta/\varepsilon)^2}\right)\right] + O(\delta)$$

$$= \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\varepsilon/\delta^2\right)}_{\ll 1 \text{ for } \varepsilon^{1/2} \ll \delta \ll 1} + O(\delta)$$

$$\therefore \quad I = I_1 + I_2 = -1 + 1/\delta + \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\frac{\varepsilon^2}{\delta^3}, \frac{\varepsilon}{\delta^2}, \delta\right)}_{\ll 1}$$

$$= \frac{\pi}{2\varepsilon} - 1 + \cdots \quad \text{as } \varepsilon \to 0$$

5. Matched Asymptotic Expansions

5.1 Singular Perturbations

Example $\qquad \varepsilon y'' + y' + y = 0 \qquad 0 < x < 1, \quad y(0) = a, \; y(1) = b.$

$\varepsilon = 0$ $\quad y' + y = 0$. Hence $y = Ae^{-x}$, which cannot satisfy both boundary conditions in general.

This is a singular perturbation problem.

More generally suppose D_ε is a differential operator that depends on a small parameter ε, e.g. $D_\varepsilon = \varepsilon \, d^2/dx^2 + d/dx + 1$.

Then a differential equation $D_\varepsilon y = 0$ with boundary conditions is a singular perturbation problem if the order of $D_0 y$ is less than the order of $D_\varepsilon y$ as $\varepsilon \to 0$

[Since the solution of $D_0 y$ cannot satisfy BCs in general].

Suppose $D_\varepsilon = \varepsilon \frac{d^k}{dx^k}$ + lower order derivatives.

* Over most of the range, $\varepsilon \frac{d^k y}{dx^k}$ is small and y satisfies $D_0 y = 0$ to good approximation.

* In some regions, typically near boundaries, $\varepsilon \frac{d^k y}{dx^k}$ is not small and y adjusts to satisfy BCs.

The usual procedure for finding a solution to a singular ODE problem is:

(*) Determine the scaling in the boundary layers e.g.

$$x = \varepsilon \hat{x} \quad \text{or} \quad x = \varepsilon^{1/2} \hat{x}$$

(*) Find the asymptotic expansions in the boundary layers ("inner" solution) and outside the boundary layers ("outer" solutions).

(*) Fix the constants of integration in these solutions by

- demanding the inner solutions satisfy the BCs
- "matching" - ensuring the expansion of the inner and outer solutions agree in an overlap region between them.

This is the method of Matched Asymptotic Expansions

Previous Example $\quad \varepsilon y'' + y' + y = 0 \quad 0 < x < 1, \quad y(0) = a, \; y(1) = b.$

Left Hand Boundary Scaling

Let $x = \varepsilon^{\alpha} x_L \quad y(x) = y_L(x_L)$ with $\alpha > 0$.

$$\therefore \quad dy/dx = \frac{1}{\varepsilon^{\alpha}} \frac{dy_L}{dx_L} \quad \text{and} \quad \varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$$

Dominant balance $\quad 1 - 2\alpha = -\alpha \quad \therefore \alpha = 1$. Hence boundary layer has width of $\mathrm{ord}(\varepsilon)$.

Right Hand Boundary Layer: Proceeds similary with $x = 1 + \varepsilon^{\beta} x_R$, $y(x) = y_R(x_R)$.
one finds $\beta = 1$.

Develop asymptotic solution

(1) Away from boundary layers (outer region), expand $y(x) \sim y_{out,0}(x) + \varepsilon y_{out,1}(x) + \cdots$
as $\varepsilon \to 0^+$ with $x, 1-x = \mathrm{ord}(1)$

(2) Left Hand Boundary. Let $x = \varepsilon x_L$ and expand

$$y(x) = y_L(x_L) \sim y_{L,0}(x_L) + \varepsilon y_{L,1}(x_L) + \cdots \quad \text{as } \varepsilon \to 0^+ \text{ with } x_L = \mathrm{ord}(1).$$

(3) Right hand boundary. Let $x = 1 + \varepsilon x_R$ and expand

$$y(x) = y_R(x_R) \sim y_{R,0}(x_R) + \varepsilon y_{R,1}(x_R) + \cdots \quad \text{as } \varepsilon \to 0^+ \text{ with } -x_R \sim \text{ord}(1)$$

Left hand boundary layer

$$\frac{d^2 y_L}{dx_L^2} + \frac{dy_L}{dx_L} + \varepsilon y_L = 0 \quad , \quad x_L > 0 .$$

$O(\varepsilon^0)$ $$\frac{d^2 y_{L,0}}{dx_L^2} + \frac{dy_{L,0}}{dx_L} = 0 , \quad x_L > 0 .$$

$O(\varepsilon^1)$ $$\frac{d^2 y_{L,1}}{dx_L^2} + \frac{dy_{L,1}}{dx_L} + y_{L,0} = 0 , \quad x_L > 0 .$$

$$\therefore \quad y_{L,0} = A_{L,0} + B_{L,0} e^{-x_L}$$

$$y_{L,1} = A_{L,1} + B_{L,1} e^{-x_L} + \left(B_{L,0} x_L e^{-x_L} - A_{L,0} x_L \right)$$

BC $y_{L,0}(0) = a$, $y_{L,1}(0) = 0$ $\quad \therefore \quad A_{L,0} + B_{L,0} = a$, $A_{L,1} + B_{L,1} = 0$.

Right hand boundary layer

$$\frac{d^2 y_R}{dx_R^2} + \frac{dy_R}{dx_R} + \varepsilon y_R = 0 \qquad x_R < 0$$

As with left hand layer

$$y_{R,0}(x_R) = A_{R,0} + B_{R,0} e^{-x_R} \qquad (x_R < 0)$$

$$y_{R,1}(x_R) = A_{R,1} + B_{R,1} e^{-x_R} + (B_{R,0} x_R e^{-x_R} - A_{R,0} x_R)$$

with $A_{R,0} + B_{R,0} = b$, $A_{R,1} + B_{R,1} = 0$

Outer region

$$\varepsilon \frac{d^2 y_{out}}{dx^2} + \frac{dy_{out}}{dx} + y_{out} = 0 \qquad 0 < x < 1$$

$O(\varepsilon^0)$ $$\frac{dy_{out,0}}{dx} + y_{out,0} = 0$$

$O(\varepsilon^1)$ $$\frac{dy_{out,1}}{dx} + y_{out,1} = -\frac{d^2 y_{out,0}}{dx^2}$$

Solve

$$y_{out,0} = A_{out,0} e^{-x}$$

$$y_{out,1} = A_{out,1} e^{-x} - A_{out,0} x e^{-x}$$

Instead of applying BCs at $x = 0, 1$, we need to match with the left and right boundary layer solutions

Idea: There is an overlap, or intermediate, region where both expansions hold and therefore be equal.

Hence Introduce an intermediate scaling, $x = \varepsilon^{\gamma}\hat{x}$ with $0<\gamma<1$. Then with $\hat{x}>0$, $\hat{x} = \mathrm{ord}(1)$

$$x = \varepsilon^{\gamma}\hat{x} \to 0, \qquad x_L = \varepsilon^{\gamma-1}\hat{x} \to \infty \qquad \text{as } \varepsilon \to 0^+.$$

Matching requires expansions to be equal as $\varepsilon \to 0^+$ with $\hat{x}>0$, $\hat{x} = \mathrm{ord}(1)$

i.e. $$y_L(\varepsilon^{\gamma-1}\hat{x}) \sim y_{out}(\varepsilon^{\gamma}\hat{x}) \quad \text{as } \varepsilon \to 0^+ \text{ with } \hat{x}>0,\ \hat{x} = \mathrm{ord}(1)$$

We have

$$y_L(\varepsilon^{\gamma-1}\hat{x}) = A_{L,0} + \underbrace{B_{L,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{exponentially small}} + O(\varepsilon)$$

$$y_{out}(\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-\varepsilon^{\gamma}\hat{x}} + O(\varepsilon) = A_{out,0}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

Same expansions $\qquad A_{L,0} = A_{out,0} \quad$ i.e. $\quad y_{L,0}(\infty) = y_{out,0}(0)$

Matching outer and right hand boundary layer

Let $x = 1 + \varepsilon^{\gamma}\hat{x}$ with $0<\gamma<1$. As $\varepsilon \to 0^+$, with $\hat{x}<0$ and $\hat{x} = \mathrm{ord}(1)$

$$y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) = A_{R,0} + \underbrace{B_{R,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\substack{\text{exponential blow}\\ \text{up as } \varepsilon\to 0^+}} + O(\varepsilon)$$

$$y_{out}(x = 1+\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-(1+\varepsilon^{\gamma}\hat{x})} + O(\varepsilon)$$

$$= \frac{A_{out,0}}{e}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

Same expansions : $B_{R,0} = 0, \quad A_{out,0} = e A_{R,0}$

$$\therefore \begin{Bmatrix} A_{L,0} + B_{L,0} = a; & A_{R,0} + B_{R,0} = b \\ A_{L,0} = A_{out,0}; & B_{R,0} = 0; \; A_{out,0} = e A_{R,0} \end{Bmatrix} \therefore \begin{Bmatrix} A_{L,0} = eb & ; \; A_{out,0} = eb \\ B_{L,0} = a - eb ; & A_{R,0} = b; \; B_{R,0} = 0 \end{Bmatrix}$$

$$\therefore \quad y_{L,0}(x_L) = eb + (a-eb)e^{-x_L} \; ; \; y_{out,0}(x) = ebe^{-x} \; ; \; y_{R,0}(x_R) = b.$$

Agreement with exact solution

Exact solution is $y(x) = A_+ e^{\lambda_- x} - A_- e^{\lambda_+ x}$ for $0 \le x \le 1$

with $A_\pm = \dfrac{a e^{\lambda_\pm} - b}{e^{\lambda_+} - e^{\lambda_-}}$, $\lambda_\pm = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

Using expansions $\lambda_+ = -1 + O(\varepsilon)$; $\lambda_- = -\frac{1}{\varepsilon} + 1 + O(\varepsilon)$ as $\varepsilon \to 0^+$

can show

$y(\varepsilon x_L) = y_{L,0}(x_L) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $x_L > 0$, $x_L = \text{ord}(1)$

$y(x) = y_{out,0}(x) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $0 < x < 1$ with $x, 1-x = \text{ord}(1)$

$y(1+\varepsilon x_R) = y_{R,0}(x_R) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $x_R < 0$ $x_R = \text{ord}(1)$.

Higher order Matching

Using the leading order solution, the first higher order solution is given by

$$y_{L,1}(x_L) = -eb\,x_L + (a-eb)\,x_L e^{-x_L} + A_{L,1} + B_{L,1} e^{-x_L}$$

$$y_{R,1}(x_R) = -b\,x_R + A_{R,1} + B_{R,1} e^{-x_R}$$

$$y_{out,1}(x) = -eb\,x e^{-x} + A_{out,1} e^{-x}$$

<u>Recall BCs</u> $\quad y_{L,1}(0)=0 \quad y_{R,1}(0)=0 \quad \therefore A_{L,1}+B_{L,1}=A_{R,1}+B_{R,1}=0$

<u>Matching Left hand boundary layer and outer region</u>

As $\varepsilon\to 0^+$ with $\hat{x}>0$ $\hat{x}=\text{ord}(1)$ where $x=\varepsilon^{\gamma}\hat{x}$, $\quad 0<\gamma<1$

$$\begin{aligned}
y_L(x_L=\varepsilon^{\gamma-1}\hat{x}) &= y_{L,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{L,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^2) \\
&= \left(eb+(a-eb)e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + \varepsilon\Big(-ebe^{\gamma-1}\hat{x}+(a-eb)\varepsilon^{\gamma-1}\hat{x}e^{-\varepsilon^{\gamma-1}\hat{x}} \\
&\qquad + A_{L,1}+B_{L,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\Big) \\
&\qquad + O(\varepsilon^2) \\
&= eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{L,1} + O(\varepsilon^2)
\end{aligned}$$

$$\begin{aligned}
y_{out}(x=\varepsilon^{\gamma}\hat{x}) &= y_{out,0}(\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(\varepsilon^{\gamma}\hat{x}) + O(\varepsilon^2) \\
&= eb\underbrace{e^{-\varepsilon^{\gamma}\hat{x}}}_{(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))} + \varepsilon\Big(-eb\varepsilon^{\gamma}\hat{x}(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})) \\
&\qquad + A_{out,1}(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))\Big) + O(\varepsilon^2)
\end{aligned}$$

$$\therefore\ y_{out}(x = \varepsilon^{\gamma}\hat{x}) = eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{out,1} + O(\varepsilon^{1+\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2})$$

need $\gamma > \frac{1}{2}$ to ensure $\varepsilon^{2\gamma}$ term subleading compared to $O(\varepsilon)$ term

<u>Same expansions</u> $\quad A_{L,1} = A_{out,1}$

<u>Note</u> some terms jump order eg. $-eb\varepsilon^{\gamma}\hat{x}$ arises from $y_{out,0}$ even though it's higher order and arises from $y_{out,1}$ in the expansion of the outer

<u>Matching Right hand boundary layer and outer</u>

- As $\varepsilon \to 0^{+}$ with $\hat{x} < 0$, $\hat{x} = \mathrm{ord}(1)$, $x_R = \varepsilon^{\gamma-1}\hat{x}$

$$y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) = y_{R,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{R,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^2)$$

$$= b + \varepsilon\left(-b\varepsilon^{\gamma-1}\hat{x} + A_{R,1} + B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + O(\varepsilon^2)$$

$$= \underbrace{\varepsilon B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{Exponentially leading term}} + b - \varepsilon^{\gamma}b\hat{x} + \varepsilon A_{R,1} + O(\varepsilon^2)$$

As $\varepsilon \to 0$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1)$, $x = 1 + \varepsilon^{\gamma}\hat{x}$

$$y_{out}(x = 1+\varepsilon^{\gamma}\hat{x}) = y_{out,0}(1+\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(1+\varepsilon^{\gamma}\hat{x}) + O(\varepsilon^2)$$

$$= ebe^{-(1+\varepsilon^{\gamma}\hat{x})} + \varepsilon\left(-eb(1+\varepsilon^{\gamma}\hat{x})e^{-(1+\varepsilon^{\gamma}\hat{x})} + A_{out,1}e^{-(1+\varepsilon^{\gamma}\hat{x})}\right) + O(\varepsilon^2)$$

$$= b(1-\varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})) + \varepsilon\left(-b(1+\varepsilon^{\gamma}\hat{x})(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})) + \frac{A_{out,1}}{e}(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))\right) + O(\varepsilon^2)$$

$$= b - \varepsilon^{\gamma} b\hat{x} - \varepsilon b + \varepsilon A_{out,1}/e + O(\varepsilon^{2\gamma}, \varepsilon^{1+\gamma}, \varepsilon^2)$$

As before, $\gamma > 1/2$.

Same expansions: $\boxed{A_{R,1} = A_{out,1}/e - b \; ; \; B_{R,1} = 0}$

Hence
$$\left\{\begin{array}{ll} \underline{\text{BCs}} & A_{L,1} + B_{L,1} = A_{R,1} + B_{R,1} = 0 \\ \underline{\text{Matching}} & A_{L,1} = A_{out,1} \; ; \; B_{R,1} = 0 \; ; \; A_{R,1} = A_{out,1}/e - b \end{array}\right\} \therefore \left\{\begin{array}{l} A_{R,1} = B_{R,1} = 0 \\ A_{out,1} = A_{L,1} = -B_{L,1} \\ \quad = eb \end{array}\right\}$$

Thus
$$y_{L,1}(x_L) = -ebx_L + (a-eb)x_L e^{-x_L} + eb(1-e^{-x_L})$$
$$y_{out,1}(x) = -ebx e^{-x} + ebe^{-x}$$
$$y_{R,1}(x) = -bx_R.$$

Note
$$\lim_{x\to 1} y_{out}(x) = \lim_{x\to 1}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = b + O(\varepsilon^2)$$
$$\lim_{x\to 0} y_{out}(x) = \lim_{x\to 0}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = eb + O(\varepsilon)$$

$\therefore$ $y_{out}(x)$ satisfies BC at $x=1$, at least to $O(\varepsilon^2)$ $\therefore$ Boundary layer not required at $x=1$.

However $\lim_{x\to 0} y_{out}(x) \neq a$ $\therefore$ Boundary layer at $x=0$ required.

Van Dyke's Matching Rule

- Using the intermediate variable $\hat{x}$ yields long calculations
- Van Dyke's matching rule is quicker and usually works :

$$\underbrace{m \text{ terms inner}\left[(n \text{ terms outer})\right]} = \underbrace{n \text{ terms outer}\left[(m \text{ terms inner})\right]}$$

5.14

n terms in the outer expansion, written in terms of the inner variable and expanded to m^{th} order in the inner variable	=	m terms in the inner expansion written in terms of the outer variable and expanded to n^{th} order in the outer variable

Example At the left hand boundary. $y_L(x_L) = A_{L,0} + (a - A_{L,0})e^{-x_L} + O(\varepsilon)$

$y_{out}(x) = A_{out,0}\, e^{-x} + O(\varepsilon)$, $\quad x = \varepsilon x_L$

LHS 1 term outer

$$= A_{out,0}\, e^{-x}$$
$$= A_{out,0}\, e^{-\varepsilon x_L}$$
$$= A_{out,0}\,(1 + O(\varepsilon x_L))$$

RHS 1 term inner

$$= A_{L,0} + (a - A_{L,0})e^{-x_L}$$
$$= A_{L,0} + (a - A_{L,0})e^{-x/\varepsilon} = A_{L,0} + \text{exponentially small}$$

$\therefore$ $A_{out,0}$ = 1 term inner [(1 term outer)] = 1 term outer [(1 term inner)] = $A_{L,0}$

$\therefore$ $A_{L,0} = A_{out,0} = eb$

using BC at $x=1$, noting there is no boundary layer there