Note This gives $\lim_{x \to 0} y_{out,0}(x) = \lim_{x_L \to \infty} y_L(x_L)$ as previously observed

Example 2^{nd} order matching

LHS. 2 term outer

$$= A_{out,0} e^{-x} + \varepsilon \left(A_{out,1} e^{-x} - A_{out,0} x e^{-x} \right)$$

$$= eb e^{-\varepsilon x_L} + \varepsilon \left(A_{out,1} e^{-\varepsilon x_L} - eb \varepsilon x_L e^{-x_L \varepsilon} \right)$$

$$= eb - \varepsilon eb x_L + \varepsilon A_{out,1} + O(\varepsilon^2)$$

RHS 2 term inner

$$= A_{L,0} + (a - A_{L,0}) e^{-x_L} + \varepsilon \left(A_{L,1} - A_{L,1} e^{-x_L} - A_{L,0} x_L + (a - A_{L,0}) x_L e^{-x_L} \right)$$

$$= eb + (a - eb) e^{-x/\varepsilon} + \varepsilon \left(A_{L,1} - A_{L,1} e^{-x/\varepsilon} - eb x/\varepsilon + (a - eb) x \varepsilon e^{-x/\varepsilon} \right)$$

$$= eb + \varepsilon (A_{L,1}) - eb x + \text{exponentially small terms.}$$

Noting $\varepsilon x_L = x$, we have $A_{L,1} = A_{out,1} = eb$

$$\therefore \quad y_{out}(x) = ebe^{-x} + \varepsilon eb(1-x)e^{-x} + \cdots$$

$$y_L(x_L) = eb + (a-eb)e^{-x_L} + \varepsilon\left(eb(1-e^{-x_L}) - eb\,x_L + (a-eb)x_L e^{-x_L}\right) + \cdots$$

Exercise repeat for 1 term inner [(2 terms outer)] = 2 terms outer [(1 term inner)]

$\boxed{\text{Warning}}$ Treat Logarithmic terms as $O(1)$ in Van Dyke's matching rule due to their size relative to powers.

Composite Expansion

Aim To combine inner and outer expansions to obtain a uniformly valid expansion (for plotting etc)

$$y_{composite} = (p \text{ terms outer}) + (p \text{ terms inner}) - \underbrace{p \text{ terms inner}\left[(p \text{ terms outer})\right]}_{p \text{ terms outer}\left[(p \text{ terms inner})\right] \text{ by Van Dyke.}} \quad p \in \mathbb{N}$$

Subtract $\quad$ p terms inner [(p terms outer)] $\quad$ as it has been counted twice in the overlap region.

Example

p=1 $\quad y_{composite}$

$$= y_{out,0}(x) + y_{L,0}(x/\varepsilon) - 1 \text{ term inner}\left[(1 \text{ term outer})\right]$$

$$= ebe^{-x} + eb + (a-eb)e^{-x/\varepsilon} - eb$$

$$= ebe^{-x} + (a-eb)e^{-x/\varepsilon}$$

p=2 $\quad y_{composite}$

$$= y_{out,0}(x) + \varepsilon y_{out,1}(x) + y_{L,0}(x/\varepsilon) + \varepsilon y_{L,1}(x/\varepsilon)$$
$$- 2 \text{ term inner}\left[(2 \text{ term outer})\right]$$

$$= ebe^{-x} + \varepsilon \cdot eb \cdot (1-x)e^{-x}$$
$$+ eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(eb(1-e^{-x/\varepsilon}) - ebx/_{\varepsilon} + (a-eb)\frac{x}{\varepsilon}e^{-x/\varepsilon}\right)$$
$$- eb + ebx - \varepsilon eb$$

$$= ebe^{-x} + (a-eb)(1+x)e^{-x/\varepsilon} - \varepsilon eb(1-x)e^{-x} - eebe^{-x/\varepsilon}$$

Choice of rescaling, revisited

In left hand boundary layer, began with scaling $x = \varepsilon^{\alpha} x_L$, $y(x) = y_L(x_L)$.

$$\varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L^2} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$$

$\alpha = 0$		Balance	Outer Solution
$0 < \alpha < 1$		Dominant	Overlap region
$\alpha = 1$	Balance		Inner Solution
$\alpha > 1$	Dominant		Sub-inner

The inner and outer solutions can be matched as they share a common term, which is dominant in the overlap region

There are two dominant balances

$$\alpha = 0 \text{ (outer)} \quad \text{and} \quad \alpha = 1 \text{ (inner)}$$

These correspond to distinguished limits in which $x = \mathrm{ord}(1)$ and $x = \mathrm{ord}(\varepsilon)$ respectively.

5.2 Where is the boundary layer?

$\boxed{\text{· For a non-trivial boundary layer, the inner solution decays on approaching the outer region.}}$

Saw this previously in the example

· **New example**

$$\varepsilon y'' + p(x) y' + q(x) y = 0 \qquad 0 < x < 1$$

$$y(0) = A \quad y(1) = B \qquad 0 < \varepsilon \ll 1$$

$$p, q \text{ smooth; } p(x) > 0$$

RH boundary layer

Let $x = 1 + \delta\hat{x}$ $\quad y(x) = y_R(\hat{x})$

as ever $'$ is derivative wrt argument.

$$\frac{\varepsilon}{\delta^2} y_R'' + \underbrace{p(1+\delta\hat{x})}_{p(1)+O(\delta)} \frac{1}{\delta} y_R' + \underbrace{q(1+\delta\hat{x})}_{q(1)+O(\delta)} y_R = 0$$

Only balance with y_R'' is between 1^{st} & 2^{nd} terms $\therefore$ $\boxed{\varepsilon = \delta}$

$$\therefore \quad y_R'' + \left[p(1) + \varepsilon\hat{x}p'(1) + \cdots\right] y_R' + \varepsilon\left[q(1) + \varepsilon\hat{x}q'(1) + \cdots\right] y_R = 0$$

With $y_R(\hat{x}) \sim y_{R,0} + \varepsilon y_{R,1} + \cdots$

$\underline{O(\varepsilon^0)}$ $\qquad y_{R,0}'' + p(1) y_{R,0}' = 0$

$$\therefore \; y_{R,0}(\hat{x}) = J + Ke^{-p(1)\hat{x}}$$

Matching $y_{R,0}(-\infty)$ with outer implies $K \equiv 0$, as we have exponential blow up.

$\therefore \; y_{R,0}(\hat{x}) = A$ and no rapid variation in boundary layer

$\therefore$ No boundary layer required.

LH Boundary layer

$$y(x) = y_L(\hat{x}), \quad x = \varepsilon \hat{x}, \qquad y_{L,0} = M + Ne^{-p(0)\hat{x}}$$

Possible to match outer solution without $N \equiv 0$, as $y_{L,0}(\infty)$ finite $\therefore$ Can have boundary layer, illustrating above statement.

Example

$$\varepsilon^2 f'' + 2f(1-f^2) = 0 \qquad |x|<1 \quad f(\pm 1) = \pm 1.$$

- Outer solution one of $f = 0, 1, -1$.
- Near LH boundary $f = -1$ OK; similarly $f = +1$ near RH boundary
- Boundary layer in interior

Let $x = x_0 + \varepsilon X \qquad f(x) = F(X)$

$$\therefore \quad F'' + 2F(1-F^2) = 0 \qquad X \in (-\infty, \infty)$$

$$F \to \pm 1 \qquad \text{as } X \to \pm\infty$$

Solution $F(X) = \tanh(X - X_*)$ ← constant.

Note x_0, X_* undetermined.

By symmetry $f(x) = -f(-x)$ as both satisfy ODE.

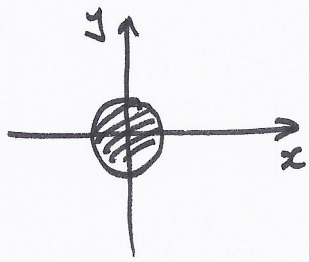
$\therefore f(0) = -f(0) \quad \therefore x_0 = X_* = 0$

$\therefore f = \tanh(x/\varepsilon)$ Agrees with exact solution

Position of transition layer exponentially sensitive to BCs. Can be analysed with WKBJ method, but beyond scope of course.

5.3 Boundary Layers in PDES

Example (2D) $\underline{u} \cdot \nabla T = \varepsilon \nabla^2 T$ for $r^2 = x^2 + y^2 > 1$



with $T = 1$ on $r = 1$ and $T \to 0$ as $r \to \infty$

$\underline{u} = \nabla \varphi, \quad \varphi = (r + 1/r)\cos\theta = x + \frac{x}{x^2+y^2}$

Outer $T \sim T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \dots$ as $\varepsilon \to 0^+$ with $r = \mathrm{ord}(1)$.

$O(\varepsilon^0)$ $\quad \underline{u} \cdot \nabla T_0 = 0, \quad T_0 \to 0$ as $r \to \infty$, $\underline{r > 1 \text{ (Outer)}}$

on a curve with $\frac{d\underline{r}}{ds} = \underline{u}$, $\frac{dT_0}{ds} = \nabla T_0 \cdot \frac{d\underline{r}}{ds} = \nabla T_0 \cdot u = 0$

curve arclength

For $r > 1$ $\quad \frac{dx}{ds} = \frac{\partial \varphi}{\partial x} = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2} = 1 - \frac{\cos 2\theta}{r^2} > 0$

$\therefore$ For $r>1$, all such curves go to infinity, where $T_0 = 0$

$\therefore T_0 = 0$ as T_0 invariant on these curves.

<u>Inner</u>

$$\left(1 - \frac{1}{r^2}\right)\cos\theta \; T_r - \left(1 + \frac{1}{r^2}\right)\frac{\sin\theta}{r} T_\theta = \varepsilon\left(T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta}\right)$$

Let $r = 1 + \delta(\varepsilon)\rho$ $\quad T(r,\theta) = \hat{T}(\rho,\theta)$ with

$\delta \to 0^+$, $\rho = \mathrm{ord}(1)$ as $\varepsilon \to 0^+$.

$$\therefore \left(1 - \frac{1}{(1+\delta\rho)^2}\right)\frac{\cos\theta}{\delta}\hat{T}_\rho - \left(1 + \frac{1}{(1+\delta\rho)^2}\right)\frac{\sin\theta}{1+\delta\rho}\hat{T}_\theta$$

$$= \frac{\varepsilon}{\delta^2}\hat{T}_{\rho\rho} + \frac{\varepsilon}{\delta(1+\delta\rho)}\hat{T}_\rho + \frac{\varepsilon}{(1+\delta\rho)^2}\hat{T}_{\theta\theta}$$

OKAY

$$\therefore \left(2\delta\rho + O(\delta^2)\right)\frac{\cos\theta}{\delta}\hat{T}_\rho - \cancel{\left(2 + O(\delta)\right)\sin\theta\,\hat{T}_\theta}$$

$$= \frac{\varepsilon}{\delta^2}\hat{T}_{\rho\rho} + \cancel{\frac{\varepsilon}{\delta}\left(1+O(\delta)\right)\hat{T}_\rho} + \cancel{\varepsilon\left(1+O(\delta)\right)\hat{T}_{\theta\theta}}$$

never going to balance

$\varepsilon/\delta^2 \sim O(1)$ Let $\delta = \varepsilon^{1/2}$

$$\hat{T} = \hat{T}_0 + \varepsilon \hat{T}_1 + \cdots$$

$$2\rho\cos\theta \frac{\partial \hat{T}_0}{\partial \rho} - 2\sin\theta \frac{\partial \hat{T}_0}{\partial \theta} = \frac{\partial^2 \hat{T}_0}{\partial \rho^2}$$

BC $\hat{T}_0 = 1$ on $\rho = 0$ with $\hat{T}_0 \to 0$ as $\rho \to \infty$ to match outer.

(Leading order Outer limit of inner is Inner limit of outer $\equiv 0$).

Seek similarity solution of form $\hat{T}_0 = f(\eta)$, $\eta = \rho g(\theta)$

$$\frac{\partial \hat{T}_0}{\partial \rho} = g f' \qquad \frac{\partial^2 \hat{T}_0}{\partial \rho^2} = g^2 f'' \qquad \frac{\partial \hat{T}_0}{\partial \theta} = \rho g' f'$$

$$\therefore \quad 2\rho\cos\theta\, g(\theta) f' - 2\sin\theta\, \rho g'(\theta) f' = g^2(\theta) f''$$

$$\therefore \quad (\rho g(\theta)) f' \left[\underbrace{\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta\, g'(\theta)}{g^3}}_{} \right] = f''$$

exercise: Show this is indeed WLOG by solving for a general negative constant and confirming the same result for $\hat{T}_0$ is found

-ve constant (if not -ve f will blow up as $\rho \to \infty$)

given g>0

WLOG Set constant $= -1$

$\therefore$ Solve $2\cos\theta\, g(\theta) - 2\sin\theta\, g'(\theta) = -g^3(\theta)$ generates error function which is well understood

Let $g = \frac{1}{p^{1/2}}$ will convert this to a linear ODE

Find $g(\theta) = \frac{|\sin\theta|}{(J + \cos\theta)^{1/2}}$ is a suitable solution.

While J is the constant of integration ... < 1 means blow up ... > 1 means T=1 for theta = Pi ... makes no sense ... upstream heated ... hence choose J= 1.

Then $f'' + \eta f' = 0 \quad \therefore \quad f = J\int_\eta^\infty e^{-u^2/2}\,du + K$

With $\hat{T}_0 = f(\eta) \to 0$ as $\rho \to \infty$, i.e. $\eta \to \infty$, $K = 0$.

$\hat{T}_0(\rho = 0) = 1 \quad \therefore \quad f(0) = 1 \quad \therefore \quad f(\eta) = \sqrt{\frac{2}{\pi}}\int_\eta^\infty e^{-u^2/2}\,du$

$\therefore$ Solution to leading order is

$$T(r,\theta) = \hat{T}(\rho,\theta) = f(\rho g(\theta))$$

$$= \sqrt{\frac{2}{\pi}}\int_{\frac{(r-1)}{\varepsilon^{1/2}}\frac{|\sin\theta|}{(1+\cos\theta)^{1/2}}}^{\infty} e^{-u^2/2}\,du$$

Solution fails for $\theta = 0$

as we do not satisfy BC ~~near~~ at infinity.

Boundary Layer at infinity, logs

$$(x^2y')' + \varepsilon x^2 y y' = 0 \qquad x>1,\ y(1)=0,\ y(\infty)=1$$
$$0<\varepsilon\ll 1$$

Try $y \sim y_0(x) + \varepsilon y_2(x) + \dots$ using boundary conditions.

$O(\varepsilon^0)$ $\quad (x^2y_0')'=0 \quad \therefore y = 1 - 1/x$

$O(\varepsilon^1)$ $\quad (x^2y_2')' = -x^2y_0y_0' = -1 + 1/x$

$\therefore$ using $y_2(1)=0$, $\quad y_2 = A(1-1/x) - \ln x - \dfrac{\ln x}{x}$

cannot satisfy $y_2(\infty)=0$ (Both $-1 + 1/x$ are homogeneous solutions to $(xf')'=0$, hence a resonant forcing occurs)

Try $x = X/\delta_1(\varepsilon)$ $\quad y = 1 + \delta_2(\varepsilon)Y(X)$ $\quad$ with $\delta_1, \delta_2 \to 0$, $X = \mathrm{ord}(1)$ as $x \to \infty$

Dominant balance $\quad \delta_2 \dfrac{d}{dX}\left(X^2\dfrac{dY}{dX}\right) + \dfrac{\varepsilon\delta_2}{\delta_1}X^2\dfrac{dY}{dX} + \dfrac{\varepsilon\delta_2^2}{\delta_1}X^2Y\dfrac{dY}{dX} = 0$

$\delta_1 = \varepsilon$, δ_2 undetermined

small "oh"

Let $Y(X) = Y_0(X) + o(1)$

$$\frac{d}{dX}\left(X^2\frac{dY_0}{dX}\right) + X^2\frac{dY_0}{dX} = 0$$

$$Y_0(X) = B\int_X^\infty \frac{e^{-s}}{s^2}\,ds \qquad \text{noting } Y_0(\infty)=0$$

$\underbrace{\text{Splitting range of integral}}_{\text{exercise}}$, $\quad Y_0(X) = B\left[1/X + \ln X + O(1)\right]$ as $X \to 0^+$

Need this limit for matching

Intermediate variables $\quad \hat{x} = \varepsilon^\alpha x = \varepsilon^{\alpha-1}X$