7. WKB Method

· After Wentzel, Kramers, Brillouin (1920s) Also known as WKBJ where J is for Jeffries. First used by Liouville and Green in [redacted] (1830s).

· Important in semi-classical analysis of quantum mechanics.

Example

$$\left.\begin{array}{l}\varepsilon^2 y'' + y = 0 \\ 0 < \varepsilon \ll 1\end{array}\right\} \Rightarrow y = R\cos\left(x/\varepsilon + \theta\right) \quad ; R, \theta \in \mathbb{R}, \text{ consts.}$$

High frequency oscillations. What if frequency of oscillation depends on the slow scale...

Example

$$\varepsilon^2 y'' + q(x) y = 0 \qquad q(x) > 0$$
$$0 < \varepsilon \ll 1$$

Try multiple scales $\quad x = \varepsilon X \quad \therefore \quad \dfrac{d^2y}{dX^2} + q(\varepsilon X) y = 0$

treat x, X as independent

$$\frac{dy}{dX} = \frac{\partial y}{\partial X} + \frac{\partial x}{\partial X}\frac{\partial y}{\partial x} = \left(\frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial x}\right) y$$

$$\therefore \quad y_{XX} + 2\varepsilon y_{xX} + \varepsilon^2 y_{xx} + q(x) y = 0$$

Let $y = y_0(x, X) + \varepsilon y_1(x, X) + \cdots$

0th order $\quad y_{0XX} + q(x)y_0 = 0 \quad \therefore \quad y_0 = R(x)\cos\left(\sqrt{q(x)}\,X + \theta(x)\right)$

1st order $\quad y_{1XX} + q(x)y_1 = -2y_{0xX}$

$$= -2\left[R(x)\cos\left(\sqrt{q(x)}\,X + \theta(x)\right)\right]_{xX}$$

$$= +2\left[\sqrt{q(x)}\,R(x)\sin\left(\sqrt{q(x)}\,X + \theta(x)\right)\right]_x$$

$$= 2\frac{\partial}{\partial x}\left[\sqrt{q(x)}\,R(x)\right]\sin\left(\sqrt{q(x)}\,X+\theta(x)\right) + 2\frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X+\theta(x)\right)\sqrt{q}\,R(x)\cos\left(\sqrt{q(x)}\,X+\theta(x)\right)$$

Both terms on RHS are resonant.

Secular conditions : $\quad \frac{\partial}{\partial x}\left(\sqrt{q(x)}\,R(x)\right) = 0 = \sqrt{q}\,R\,\frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X + \theta(x)\right)$

$q>0 \;\therefore$ Either $R(x) = 0$ (trivial solution obvious and useless)

or $\quad \frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X + \theta(x)\right) = 0 \quad$ i.e. $\quad \frac{\partial}{\partial x}\left(\sqrt{q}\right)X + \theta'(x) = 0$

i.e. $$X = -\frac{\theta'(x)}{\partial/\partial x\left(\sqrt{q}\right)}$$ ✱ function of x cannot be equal to X for all X.

Happens whenever frequency of fast oscillation drifts on a slow scale.

$\therefore$ Try a WKB expansion $\quad y = \exp\left[\frac{i}{\varepsilon}\varphi(x)\right]A(x,\varepsilon)$

$$y' = e^{i\varphi/\varepsilon}\left[i\varphi' A/_{\varepsilon} + A' \right]$$

$$y'' = e^{i\varphi/\varepsilon}\left[\frac{i\varphi'}{\varepsilon}\left(i\varphi' A/_{\varepsilon} + A'\right) + \left(i\varphi' A/_{\varepsilon} + A'\right)' \right]$$

$$\therefore \quad \varepsilon^2 e^{i\varphi/\varepsilon}\left[\frac{-\varphi'^2 A}{\varepsilon^2} + \frac{2i\varphi' A' + i\varphi'' A}{\varepsilon} + A'' \right] + q e^{i\varphi/\varepsilon} A = 0$$

$$\therefore \quad \varepsilon^2 A'' + \{2i\varepsilon\varphi' A'\} + \{-\varphi'^2 + i\varepsilon\varphi'' + q\} A = 0$$

Let $A = A_0 + \varepsilon A_1 + \cdots$

$O(\varepsilon^0)$ $\quad \{-\varphi'^2 + q\} A_0 = 0 \quad \therefore$ For $A_0 \not\equiv 0$, $\quad \underline{\underline{\varphi'^2 = q}}$

$O(\varepsilon^1)$ $\quad 2i\varphi' A_0' + i\varphi'' A_0 + \underbrace{\{-\varphi'^2 + q\}}_{0} A_1 = 0$

$$\therefore \quad \frac{2A_0'}{A_0} + \frac{\varphi''}{\varphi'} = 0$$

$$\therefore \quad \log A_0^2 \varphi' = \text{Const} \quad \therefore \quad A_0 = \frac{\alpha_0}{(\varphi')^{1/2}} \qquad \alpha_0 \in \mathbb{C}$$

$O(\varepsilon^{n+1})$ $\quad A_{n-1}'' + 2i\varphi' A_n' + i\varphi'' A_n = 0$

$$\therefore \quad \left((\varphi')^{1/2} A_n\right)' = -\frac{1}{2i(\varphi')^{1/2}} A_{n-1}''$$

$$\therefore \quad A_n = \frac{i}{2(\varphi')^{1/2}} \int^x \frac{A_{n-1}''(s)}{2(\varphi')^{1/2}(s)}\, ds$$

At leading order

$$y \sim \frac{\alpha_+}{(q(x))^{1/4}} \exp\left[\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right] + \frac{\alpha_-}{(q(x))^{1/4}} \exp\left[-\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right]$$

In principle can go to higher orders as A_n generally known.

$\alpha_\pm \in \mathbb{C}$.

Method breaks down near $\varphi' = 0$, as amplitude blows up.

Fix at turning points considered later.

Example

Find eigenvalues with $\lambda \gg 1$ for $p(x)$ a positive function and

$$y'' + \lambda p(x) y = 0 \qquad 0 < x < 1 \qquad y(0) = 0 \quad y(1) = 0$$

Let $\lambda = 1/\varepsilon^2$, $0 < \varepsilon \ll 1$. Then

$$\varepsilon^2 y'' + p(x) y = 0$$

WKB Let $y = e^{i\varphi/\varepsilon} A(x,\varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$.

$O(\varepsilon^0)$ $\quad \varphi'^2 = p \quad \therefore \varphi' = \pm\sqrt{p(x)} \quad \therefore \varphi = \pm \int_0^x p(s)^{1/2}\, ds$

const of integration absorbed into A_0.

$O(\varepsilon^1)$ $\quad 2\varphi' A_0' + \varphi'' A_0 = 0 \quad \therefore A_0 = \dfrac{\text{const}}{(p(x))^{1/4}}$

Two lin. independent solutions

$$y_+ \sim A_0 e^{i\varphi/\varepsilon} \qquad y_- \sim A_0 e^{-i\varphi/\varepsilon}$$

General solution, at leading orders:

$$y \sim \alpha A_0(x) \cos\left(\varphi(x)/\varepsilon\right) + \beta A_0(x) \sin\left(\frac{\varphi(x)}{\varepsilon}\right)$$

$$\alpha, \beta \in \mathbb{R}.$$

small "oh"

$y(0) = 0 \quad \therefore \alpha = 0$

$y(1) = 0$ satisfied at leading order only if $\beta A_0(1) \sin\left(\frac{\varphi(1)}{\varepsilon}\right) = o(1)$

We have $A_0(1) \neq 0$, $\beta > 0$ for a non-trivial solution

$$\therefore \varphi(1) \sim n\pi\varepsilon \quad \text{as } \varepsilon \to 0.$$

$$\therefore \frac{1}{\sqrt{\lambda_n}} = \varepsilon_n \sim \frac{\varphi(1)}{n\pi} = \frac{1}{n\pi}\int_0^1 \sqrt{p(x)}\,dx$$

n^{th} eigenvalue

$$\therefore \lambda_n \sim \left(\frac{n\pi}{\int_0^1 \sqrt{p(x)}\,dx} \right)^2 \quad \text{as } n \to \infty$$

Example **Semi-Classical Quantum. Turning Points.**

The non-dimensional steady state Schrödinger equation for the even wave-functions of the simple harmonic oscillator is given by

$$\psi'' - x^2\psi = -E\psi$$
$$\psi \to 0 \text{ as } x \to \infty, \quad \psi'(0) = 0.$$

Find the large, $E \gg 1$, energy eigenvalues.

Let $y = \psi$. $x = \bar{x}/\sqrt{\varepsilon}$ with $\varepsilon = 1/E$. Then, dropping bars,

$$\varepsilon^2 y'' + (1 - x^2) y = 0$$
$$y(\infty) = 0, \; y'(0) = 0, \quad 0 < \varepsilon \ll 1.$$

Let $y = e^{i\varphi/\varepsilon} A(x, \varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$

WKB $O(\varepsilon^0)$ $\varphi' = \pm\sqrt{1 - x^2}$ $O(\varepsilon^1)$ $A_0 = \dfrac{\text{const}}{(1 - x^2)^{1/4}}$

Hence

For $0<x<1$, $$y \sim \frac{M_0}{(1-x^2)^{1/4}} e^{1/\varepsilon \int_0^x \sqrt{1-s^2}\, ds} + \frac{N_0}{(1-x^2)^{1/4}} e^{-1/\varepsilon \int_0^x \sqrt{1-s^2}\, ds}$$

using $y'(0)=0$

$$\sim \frac{P_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\, ds\right)$$

For $x>1$ $$y \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds}$$ using $y(\infty)=0$

However, these breakdown near $x \simeq 1$ as $\varphi'(1)=0$.

Resolve using matched asymptotics

Inner region around $x=1$ Let $x = 1+\delta_1(\varepsilon)X$ $Y(X) = \delta_2(\varepsilon)\, y(x)$

$$\frac{\varepsilon^2}{\delta_1^2}\frac{d^2Y}{dX^2} + \Big(1-\underbrace{(1+2\delta_1 X + \delta_1^2 X^2)\Big)Y}_{2\delta_1 XY + \delta_1^2 X^2 Y} = 0$$

Dominant balance when $2\delta_1^3 = \varepsilon^2$ $\therefore$ Let $\delta_1 = \dfrac{\varepsilon^{2/3}}{2^{1/3}}$.

δ_2 undetermined as yet

small "oh"

With $Y = Y_0(X) + o(1)$

$$\frac{d^2 Y_0}{dX^2} - XY_0 = 0 \quad \therefore \quad Y_0 = R_0\,\mathrm{Ai}(X) + S_0\,\mathrm{Bi}(X)$$ where Ai, Bi are Airy functions.

<u>Airy Functions</u>

$$\mathrm{Ai}(X) = \frac{1}{\pi}\int_0^\infty \cos\left(t^3/3 + Xt\right)dt \sim \frac{1}{2\sqrt{\pi}\,X^{1/4}}\,e^{-2/3\,X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}}\sin\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty.$$

$$\mathrm{Bi}(X) = \frac{1}{\pi}\int_0^\infty \exp\left(-t^3/3 + Xt\right)dt \sim \frac{1}{\sqrt{\pi}\,X^{1/4}}\,e^{2/3\,X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}}\cos\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty$$

<u>Matching</u> Inner ($X \to \infty$) with RH outer ($x \to 1^+$)

$S_0 = 0$ else Y_0 blows up as $X \to \infty$.

On matching everything scales with $\frac{1}{X^{1/4}} e^{-\frac{2}{3}X^{3/2}}$ whether using Van Dyke or intermediate region. Naively one gets simply $0=0$. Thus, on matching, insist the coefficients in front of $\frac{1}{X^{1/4}} e^{-\frac{2}{3}X^{3/2}}$ match.

<u>Matching</u> (intermediate variable)

Let $x-1=\delta_1^{\beta}\hat{x} = \delta_1 X$ $\quad (0<\beta<1)$ $\quad$ with $\hat{x} = \mathrm{ord}(1)$, $x\to 1$, $X\to\infty$, $\hat{x}>0$.

$$Y_0 = R_0 \,\mathrm{Ai}\left(\frac{\hat{x}}{\delta_1^{1-\beta}}\right) \sim \frac{R_0}{2\sqrt{\pi}}\,\frac{(\delta_1^{1-\beta})^{1/4}}{\hat{x}^{1/4}}\exp\left[-\frac{2}{3}\cdot\frac{1}{(\delta_1^{1-\beta})^{3/2}}\,\hat{x}^{3/2}\right]$$

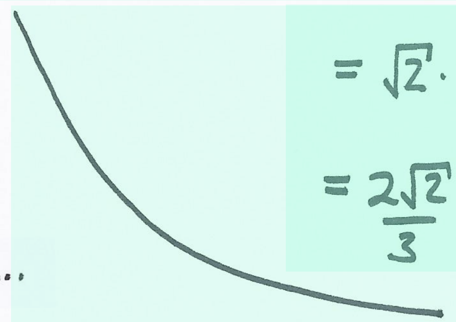
$$y \sim \frac{Q_0}{[(x-1)(x+1)]^{1/4}}\exp\left[-\frac{1}{\varepsilon}\int_1^x\sqrt{s^2-1}\,ds\right]$$

$s^2-1=(s-1)(s+1)$, $\quad s=1+\eta$

$$\int_1^x\sqrt{s^2-1}\,ds = \int_0^{x-1}\eta^{1/2}\,2^{1/2}\sqrt{1+\eta/2}\,d\eta$$

$$= \sqrt{2}\cdot\frac{2}{3}(x-1)^{3/2}+\dots$$

$$= \frac{2\sqrt{2}}{3}\,\delta_1^{3\beta/2}\,\hat{x}^{3/2}+\dots$$



$$\therefore \frac{1}{\varepsilon}\int_1^x\sqrt{s^2-1}\,ds = \frac{1}{(2^{1/3}\delta_1)^{3/2}}\,\frac{2\sqrt{2}}{3}\,\delta_1^{3\beta/2}\,\hat{x}^{3/2}+\dots$$

$$\therefore\ y \sim \frac{Q_0}{2^{1/4}\delta_1^{\beta/4}\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots$$

$$\therefore\ Y = \delta_2 y \sim \frac{Q_0\,\delta_2(\varepsilon)}{2^{1/4}(\delta_1)^{\beta/4}\hat{x}^{1/4}} \exp\left[-2/3\,\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots \sim \frac{R_0\,\delta_1^{1/4}}{2\sqrt{\pi}}\frac{1}{\delta_1^{\beta/4}}\frac{1}{\hat{x}^{1/4}}\exp\left[-\frac{2}{3}\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}\right]$$

$$\therefore\quad \delta_2 = \delta_1^{1/4} = \left(\varepsilon^{2/3}/2^{1/3}\right)^{1/4} = \frac{1}{2^{1/12}}\varepsilon^{1/6} \quad \text{and} \quad Q_0 = \frac{1}{2^{3/4}\sqrt{\pi}}R_0$$

<u>Matching</u> inner $(X \to -\infty)$ with LH outer $(x \to 1^-)$.

Let $x - 1 = \delta_1^{\gamma}\hat{x} = \delta_1 X \quad (0<\gamma<1)$ with $\hat{x} = \text{ord}(1)$, $x \to 1$, $X \to -\infty$, $\hat{x} < 0$.

$$Y_0 = R_0\,Ai\left(\frac{\hat{x}}{\delta_1^{1-\gamma}}\right) \sim \frac{R_0(\delta_1)^{\frac{1-\gamma}{4}}}{\sqrt{\pi}(-\hat{x})^{1/4}}\sin\left(\frac{2}{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}} + \frac{\pi}{4}\right)$$

$$y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}}\cos\left(\pi/4\varepsilon - \frac{1}{\varepsilon}\int_x^1\sqrt{1-s^2}\,ds\right) \qquad \text{using } \int_0^1\sqrt{1-s^2}\,ds = \pi/4$$

$$\therefore\; y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\cdot\frac{2\sqrt{2}}{3}(1-x)^{3/2} + \dots\right)$$

← Substituting $s = 1-\eta$ in integral and using $\sqrt{1-s^2} = \eta^{1/2}(2^{1/2} + O(\eta))$

$$\sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \underbrace{\frac{2\sqrt{2}}{3\varepsilon}\delta_1^{3\gamma/2}}_{\frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}}(-\hat{x})^{3/2} + \dots\right)$$

$$\sim \delta_2^{-1} Y_0 = \frac{R_0}{\sqrt{\pi}\,(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \sin\left(\pi/4 + 2/3(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}\right)$$

With $w = 2/3(-\hat{x})^{3/2}\dfrac{1}{(\delta_1^{1-\gamma})^{3/2}}$,

$$\therefore\; \frac{P_0}{2^{1/4}}\left[\cos\frac{\pi}{4\varepsilon}\cos w + \sin(\pi/4\varepsilon)\sin w\right] \sim \frac{R_0}{\sqrt{\pi}}\left[\sin\pi/4\cos w + \cos\pi/4\sin w\right]$$

$$\therefore\; \frac{P_0}{2^{1/4}}\cos\frac{\pi}{4\varepsilon} \sim \frac{R_0\sin\pi/4}{\sqrt{\pi}}, \qquad \frac{P_0\sin(\pi/4\varepsilon)}{2^{1/4}} \sim \frac{R_0}{\sqrt{\pi}}\cos\pi/4$$

For $P_0, R_0 \neq 0$ $\quad \tan\left(\frac{\pi}{4\varepsilon}\right) \sim \cot(\pi/4) = 1$ as $\varepsilon \to 0$

$$\therefore \quad \frac{\pi}{4\varepsilon} \sim \frac{\pi}{4} + n\pi \quad \text{as } n \to \infty, \quad \text{with } n \in \mathbb{N}$$

$$\therefore \quad E_n = \frac{1}{\varepsilon_n} = 1 + 4n \quad \text{as } n \to \infty, \text{ for the energy levels.}$$

<u>Once this holds</u> $\quad \cos\left(\frac{\pi}{4\varepsilon}\right) \sim \cos(\pi/4 + n\pi) = \frac{1}{\sqrt{2}}(-1)^n$

$$\left.\begin{aligned} \therefore \quad P_0 &= \frac{2^{1/4}}{\sqrt{\pi}}(-1)^n R_0 \\ &= 2 \, (-1)^n Q_0 \end{aligned}\right\} \text{Connection formula}$$

$$y_n \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon_n \int_1^x \sqrt{s^2-1}\, ds} \qquad x > 1, \; x \not\approx 1$$

$$\sim \frac{2^{1/12}}{\varepsilon^{1/6}} \cdot 2^{3/4} \cdot \sqrt{\pi}\, Q_0 \, \mathrm{Ai}\left(\frac{2^{1/3}(x-1)}{\varepsilon_n^{2/3}}\right) \qquad x \simeq 1$$

$$\sim \frac{2(-1)^n Q_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon_n}\int_0^x \sqrt{1-s^2}\, ds\right) \quad \begin{array}{l} x<1 \\ x \not\approx 1 \end{array}, \quad \varepsilon_n = \frac{1}{1+4n}, \quad n \gg 1.$$