C4.8 Complex Analysis: conformal maps and geometry

Sheet 0

This problem sheet is a companion to the Prerequisites section. The goal of these problems is to revise some standard complex analysis and get up to speed. The solutions to all of these problems are available on the course page.

Problem 1. (See also Proposition 0.1)

Let f_n be the analytic functions on a domain Ω which converge to f uniformly on every compact $K \subset \Omega$. Let $z_0 \in \Omega$. Show that $f_n'(z_0) \to f'(z_0)$.

Problem 2.

The Fundamental Theorem of Algebra which states that every polynomial of degree $n \geq 1$ has exactly n complex roots (counting multiplicity). It is a simple algebraic fact that to prove this theorem it is sufficient to show that every polynomial has at least one root. The full statement then follows by induction. In this problem we will give several proofs of this fact using different approaches.

Prove that every polynomial of degree at least 1 has at least one complex root by using

(1) Liouville theorem (Theorem 0.2)
(2) Maximum modulus principle (Proposition 0.3)
(3) Argument principle (Theorem 0.7)
(4) Rouché theorem (Theorem 0.8)

Problem 3.

Let f be analytic in a neighbourhood of infinity. That is there is $R > 0$ such that f is analytic in $\Omega = \{z : R < |z| < \infty\}$. We also assume that f is finite in this annulus. By Laurent series theorem we can write

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n, \quad R < |z| < \infty.$$

Let us consider f as a $\mathbb{C}$-valued function. Show that infinity is a removable singularity of f if and only if $a_n = 0$ for all $n \geq 1$.

On the other hand, if we consider f as a $\widehat{\mathbb{C}}$-valued function, then infinity is a removable singularity if and only if only finitely many non-zero a_n for $n \geq 1$.

Let us additionally assume that f is one-to-one and $f(z) \to \infty$ as $z \to \infty$. Show that in this case f is of the form

$$f(z) = a_1 z + \sum_{-\infty}^{0} a_n z^n,$$

where $a_1 \neq 0$.

Problem 4.

Let z_1, z_2, z_3, and z_4 are four distinct points on the complex sphere. Show that their cross-ratio is positive if and only if they either collinear or concyclic.