Question 1. Prove that $\pi(X) \leqslant 2\pi(X/2)$ for X sufficiently large.

Solution 1. By the PNT, $\pi(x) = x/\log x + x/(\log x)^2 + O(x/(\log x)^3)$, so

$$2\pi(X/2) = \frac{X}{\log X - \log 2} + \frac{X}{\log^2 X} + O(\frac{X}{\log^3 X}).$$

But

$$\frac{X}{\log X - \log 2} = \frac{X}{\log X}\Big(1 + \frac{\log 2}{\log X} + O(\frac{1}{\log^2 X})\Big).$$

which is bigger than the expression for $\pi(X)$ when X is large enough.

Question 2. Let p_n denote the n^{th} prime. Prove that

$$p_n = n\log n + n\log\log n + O(n).$$

Solution 2. Since $n = \pi(p_n)$ and $\pi(x) \sim x/\log x$, we have $n\log n = \pi(p_n)\log(\pi(p_n)) \sim p_n$ so $p_n = (1+o(1))n\log n$. In particular, $\log p_n = \log n + \log\log n + o(1)$. Therefore

$$n = \pi(p_n) = \frac{p_n}{\log p_n} + O\Big(\frac{p_n}{(\log p_n)^2}\Big) = \frac{p_n}{\log n + \log\log n + o(1)}\Big(1 + O\Big(\frac{1}{\log n}\Big)\Big)$$

This gives

$$\begin{aligned} p_n &= (n\log n + n\log\log n + o(n))(1 + O(1/\log n)) \\ &= n\log n + n\log\log n + O(n). \end{aligned}$$

Question 3. (i) Let $\theta \in (0,1)$ be such that $\Re(\rho) \le \theta$ for all non-trivial zeros ρ. Deduce that for all $x \ge 2$

$$\sum_{n<x} \Lambda(n) = x + O(x^\theta(\log x)^2).$$

(ii) Let $\gamma \in (0,1)$ be such that for all $x \ge 2$

$$\sum_{n<x} \Lambda(n) = x + O(x^\gamma).$$

Show that $\Re(\rho) \le \gamma$ for all zeros of $\zeta(s)$.
(Hint: Use partial summation to prove analytic continuation of ζ'/ζ)

(iii) Let $\alpha \in (0,1)$ be fixed. Show that if for all $x \ge 2$ we have

$$\sum_{n<x} \Lambda(n) = x + O\Big(x^\alpha \exp(\sqrt{\log x})\Big)$$

then in fact

$$\sum_{n<x} \Lambda(n) = x + O(x^\alpha(\log x)^2).$$

Solution 3. (i) This is the same as the proof of the error term under the Riemann Hypothesis from lectures. Using the explicit formula with $T = x$ gives

$$\sum_{n<x} \Lambda(n) = x - \sum_{|\rho|<x} \frac{x^\rho}{\rho} + O(\log x)^2$$

$|x^\rho| < x^\theta$ for all ρ by assumption, and $|\rho| \gg 1$ since $\zeta(0) \neq 0$. If $|\Im(\rho)| \in [n, n+1]$ then $1/|\rho| \ll 1/(n+1)$. Thus

$$\sum_{n<x} \Lambda(n) = x + O\Big(x^\theta \sum_{n\geq 0} \frac{\#\{\rho : |\Im(\rho)| \in [n, n+1]\}}{n+1}\Big) + O(\log x)^2.$$

We know that the size of the set above is $O(\log x)$, and so we find that

$$\sum_{n<x} \Lambda(n) = x + O\Big(x^\theta \sum_{n\geq 0} \frac{\log x}{n+1}\Big) + O(\log x)^2 = x + O(x^\theta (\log x)^2).$$

(ii) By partial summation for $\Re(s) > 1$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -s \int_1^\infty \frac{\sum_{m<t} \Lambda(m)}{t^{s+1}} dt$$
$$= \frac{-s}{1-s} - s \int_1^\infty \frac{\sum_{m<t} \Lambda(m) - t}{t^{s+1}} dt$$

We see that since the numerator of the integrand is $O(t^\gamma)$ by assumption, the integral converges absolutely for $\Re(s) > \gamma$. This gives an analytic continuation of ζ'/ζ to $\Re(s) > \gamma$, and so $\zeta(s)$ cannot have any zeros in this region. Hence $\Re(\rho) \leq \gamma$ for all zeros ρ.

(iii) By (ii) we have that for any $\epsilon > 0$ all zeros ρ have $\Re(\rho) \leq \alpha + \epsilon$. This means that $\Re(\rho) \leq \alpha$ for all zeros ρ. But then by (i) we see that this means that the error term can actually be taken as $x^\alpha (\log x)^2)$.

Question 4. (i) Show that for $\Re(s) > 1$ we have

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

(ii) Show that $3 + 4\cos(\theta) + \cos(2\theta) \geq 0$.

(iii) Using (i) and (ii), show that for $\sigma > 1$

$$3 \log \zeta(\sigma) + 4\Re \log \zeta(\sigma + it) + \Re\log(\zeta(\sigma + 2it) \geq 0.$$

Deduce from this that for $\sigma > 1$

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$

(iv) Deduce from the above inequality that $\zeta(1+it) \neq 0$.
(Hint: Consider $\sigma \to 1$)

Solution 4. (i) For $\Re(s) > 1$ we have

$$\zeta(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}.$$

Since $|p^s| < 1$ when $\Re(s) > 1$, we have that p^{-s} is in the convergent region of the Taylor expansion $\log(1-x) = -x - x^2/2 - x^3/3 - \dots$. Therefore

$$\log \zeta(s) = \sum_p \log\Big(1 - \frac{1}{p^s}\Big)^{-1} = \sum_p \sum_m \frac{1}{mp^{ms}}.$$

(ii) Since $\cos(2\theta) = 2\cos(\theta)^2 - 1$, we have $3 + 4\cos(\theta) + \cos(2\theta) = 2(\cos(\theta) + 1)^2 \geq 0$.

(iii) We see from (i) that

$$\begin{aligned}
&3\log\zeta(\sigma) + 4\Re\log\zeta(\sigma+it) + \Re\log(\zeta(\sigma+2it) \\
&= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}}\Big(3 + 4\Re(p^{-imt}) + \Re(p^{-2imt})\Big) \\
&= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}}\Big(3 + 4\cos(mt\log p) + \cos(2imt\log p)\Big)
\end{aligned}$$

By (ii) this is a sum of non-negative terms. Exponentiating both sides gives $\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq 1$.

(iv) Assume that $\zeta(1+it_0) = 0$ for some $t_0 \neq 0$. Then for $\sigma > 1$, we have $\zeta(\sigma+it_0) = O(\sigma-1)$. We know that for $\sigma < 2$ we have $\zeta(\sigma) \ll 1/(\sigma-1)$. Finally, ζ has no poles with $\Re(s) \geq 1$ except at $s = 1$, so $\zeta(\sigma+2it_0) \ll 1$. But then

$$1 \leq \zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \ll \frac{(\sigma-1)^4}{(\sigma-1)^3} \ll \sigma - 1.$$

Letting $\sigma \to 1$ we get a contradiction.

Question 5. It is a fact that

$$\sum_{n<x} \Lambda(n) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}).$$

where the sum is understood to be the limit as $T \to \infty$ of $\sum_{|\Im(\rho)| \leq T} x^\rho/\rho$ over non-trivial zeros ρ (it is not absolutely convergent).

(i) Using this fact, show that $\zeta(s)$ must have at least one non-trivial zero.

(ii) Show that if ρ is a non-trivial zero of $\zeta(s)$, then so is $1-\rho$.

(iii) Let $\epsilon > 0$. Using Question 3, deduce that we cannot have for all $x \geq 2$ that

$$\sum_{n<x} \Lambda(n) = x + O(x^{1/2-\epsilon}).$$

Solution 5. (i) If $\zeta(s)$ has no non-trivial zeros, then we would find that

$$\sum_{n<x} \Lambda(n) = x - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}).$$

But the right hand side is continuous, whereas the left hand side is not continuous at primes, so this is impossible.

(ii) If ρ is a zero of $\zeta(s)$, then so is $1-\rho$ by the functional equation.

(iii) We know $\zeta(s)$ must have a non-trivial zero ρ, and so also have $1-\rho$. But we have $\max(\Re(\rho), \Re(1-\rho)) \geq 1/2$, so one has real part at least $1/2$. But by Question 3, this means we cannot have $\sum_{n<x} \Lambda(n) = x + O(x^{1/2-\epsilon})$, since then all zeros would have real part at most $1/2-\epsilon$.

Question 6. Recall from Sheet 3 Q6: If $f : \mathbb{R} \to \mathbb{C}$ is smooth and non-zero only on some interval $[\alpha, \beta] \subseteq (0, \infty)$ then for any $\sigma \in \mathbb{R}$

$$f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\frac{y^s}{n^s}ds,$$

where $F(s) = \int_0^\infty f(x)x^{s-1}dx$ is a smooth function with $|F(\sigma+it)| \ll 1/|t|^{100}$ and with no zeros or poles.

(i) Show that

$$\sum_{n=1}^{\infty} \Lambda(n) f\Big(\frac{n}{y}\Big) = \frac{-1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\frac{\zeta'(s)}{\zeta(s)}ds.$$

(ii) Deduce that

$$\sum_{n=1}^{\infty} \Lambda(n) f\Big(\frac{n}{y}\Big) = y\int_0^\infty f(t)dt - \sum_{\rho} y^\rho F(\rho) + O(y^{-1/4})$$

where $\sum_\rho$ is a sum over all non-trivial zeros of $\zeta(s)$ with multiplicity.

Solution 6. (i) We choose $\sigma = 2$ and us the Mellin inversion formula for f:

$$\sum_{n=1}^{\infty} \Lambda(n) f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} \Lambda(n)\int_{2-i\infty}^{2+i\infty} \frac{y^s F(s)}{n^s}ds.$$

Since $|F(s)| \ll 1/|s|^2$ and $|n^{-s}| \ll 1/n^2$, the above converges absolutely so we may change the order of summation and integration. Since $-\zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}$, this gives the result.

(ii) As in Lectures, we may choose $T \in [T_1, T_1+1]$ such that no zeros ρ have $||\Im(\rho) - T| \leq 1/(\log T)^2$ if T_1 is large enough. By Cauchy's residue theorem

$$\frac{1}{2\pi i}\Big(\int_{-1/4+iT}^{-1/4-iT} + \int_{-1/4-iT}^{2-iT} + \int_{2-iT}^{2+iT} + \int_{2+iT}^{-1/4+iT}\Big) y^s F(s) \frac{\zeta'(s)}{\zeta(s)} ds = -yF(1) + \sum_\rho y^\rho F(\rho),$$

since $\zeta'/\zeta(s)$ has a simple pole at $s=1$ with residue -1, and simple poles at $s=\rho$ with residue m_ρ (multiplicity of zero), and $y^s F(s)$ has no poles. We have $F(s) \ll 1/|s|^{100}$ and $|y^s| = y^{\Re(s)}$. Since we stay away from zeos of $\zeta(s)$, we have that $\zeta'(s)/\zeta(s) \ll \log(2+|s|)^3$ along all the contours above. We see that the first contour contributes

$$\ll y^{-1/4} \int_{-T}^{T} \frac{\log(2+|t|)^3}{(1/4+t)^{100}} dt \ll y^{-1/4}$$

The second and fourth contours contribute $\ll y^2 (\log T)^3 / T^{100}$. Thus, letting $T \to \infty$, we see that

$$\int_{2-i\infty}^{2+i\infty} y^s F(s) \frac{\zeta'(s)}{\zeta(s)} ds = -yF(1) + \sum_\rho y^\rho F(\rho) + O(y^{-1/4}).$$

Using (i), this gives the result, noting that $F(1) = \int_0^\infty f(t) dt$.

Question 7. For this question you may use the following fact: $|\zeta(\sigma+it)| \ll |t|^{1-\sigma} \log|t|$ for $\sigma \leq 1$ and $|t| \geq 1$.

(i) Show using Perron's formula that for $2 \leq T \leq 2x$

$$\sum_{n<x} \frac{\mu(n)^2 \phi(n)}{n} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \zeta(s) Z(s) ds + O\Big(\frac{x(\log x)^3}{T}\Big),$$

where $c = 1 + 1/\log x$ and for $\Re(s) > 1$

$$Z(s) = \prod_p \Big(1 - \frac{1}{p^{2s}} - \frac{1}{p^{s+1}} + \frac{1}{p^{2s+1}}\Big).$$

(ii) Show that the product of for $Z(s)$ converges absolutely for $\Re(s) > 1/2$.

(iii) Let $\epsilon = 1/1000$. By moving the line of integration to $\Re(s) = 1/2 + \epsilon$, show that

$$\sum_{n<x} \frac{\mu(n)^2 \phi(n)}{n} = x \prod_p \Big(1 - \frac{2}{p^2} + \frac{1}{p^3}\Big) + O\Big(x^{1/2+\epsilon} T^{1/2-\epsilon} \log x\Big) + O\Big(\frac{x(\log x)^3}{T}\Big)$$

(iv) Deduce that

$$\sum_{n<x} \frac{\mu(n)^2 \phi(n)}{n} = x \prod_p \Big(1 - \frac{2}{p^2} + \frac{1}{p^3}\Big) + O\Big(x^{2/3+\epsilon}\Big).$$

Solution 7. (i) Since $\phi(n)/n < 1$, by Perron's formula

$$\sum_{n<x} \frac{\mu(n)^2\phi(n)}{n} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}G(s)ds + O\Big(\frac{x(\log x)^3}{T}\Big),$$

where for $\Re(s) > 1$

$$G(s) = \sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n^{1+s}} = \prod_p \Big(1 + \frac{p-1}{p^{1+s}}\Big).$$

We see that

$$\Big(1 + \frac{p-1}{p^{1+s}}\Big)\Big(1 - \frac{1}{p^s}\Big) = 1 + \frac{1}{p^{1+2s}} - \frac{1}{p^{1+s}} - \frac{1}{p^{2s}}.$$

Therefore, dividing through by $1 - p^{-s}$ and taking the product over p, we find for $\Re(s) > 1$ that $G(s) = \zeta(s)Z(s)$.

(ii) For $1 \geq \sigma = \Re(s) > 1/2$ we have $|1-p^{-2s}-p^{-1-s}+p^{-1-2s}| \leq 1+3p^{-2\Re(s)})$. Therefore

$$\prod_p \Big|1+\frac{1}{p^{1+2s}}-\frac{1}{p^{1+s}}-\frac{1}{p^{2s}}\Big| \leq \exp\Big(\sum_p \log(1+3p^{-2\sigma})\Big) \leq \exp\Big(\sum_p O(p^{-2\sigma})\Big).$$

This converges for $\Re(s) > 1/2$.

(iii) Since $Z(s)$ converges for $\Re(s) > 1/2$, we see that $x^sZ(s)/s$ has no poles in the region $\Re(s) \geq 1/2 + \epsilon$, and $\zeta(s)$ just has a simple pole at $s = 1$ (with residue 1). Therefore, by Cauchy's residue theorem

$$\frac{1}{2\pi i}\Big(\int_{1/2+\epsilon+iT}^{1/2+\epsilon-iT} + \int_{1/2+\epsilon-iT}^{c-iT} + \int_{c-iT}^{c+iT} + \int_{c+iT}^{1/2+\epsilon+iT}\Big)\frac{x^s}{s}\zeta(s)Z(s)ds = xZ(1).$$

Throughout the region of integration $Z(s) = O(1)$. In the second and fourth integrals, $|\zeta(s)| \ll T^{1-\sigma}\log T$, $|x^s| \ll x^\sigma$ and $|1/s| \ll 1/T$. Therefore, since $T \ll x$, these integrals contribute

$$\ll \log x \int_{-1/4}^{c} \Big(\frac{x}{T}\Big)^\sigma d\sigma \ll \frac{x^c \log x}{T^c} \ll \frac{x\log x}{T}.$$

In the first integral, $\zeta(s) \ll (1+|t|)^{1/2-\epsilon}\log x$, $|x^s| \ll x^{1/2+\epsilon}$ and $|1/s| \ll 1/(1+|t|)$. Therefore this integral contributes

$$\ll x^{1/2+\epsilon}\log x \int_{-T}^{T} \frac{dt}{(1+|t|)^{1/2+\epsilon}} \ll x^{1/2+\epsilon}T^{1/2-\epsilon}\log x.$$

Putting this together, and noting $Z(1) = \prod_p (1 - 2/p^2 + 3/p^3)$ gives the result.

(iv) Choose $T = x^{1/3}$ in for part (iii)

Question 8. Recall from Sheet 3 Q6: If $f : \mathbb{R} \to \mathbb{C}$ is smooth and non-zero only on some interval $[\alpha, \beta] \subseteq (0, \infty)$ then for any $\sigma \in \mathbb{R}$

$$f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\frac{y^s}{n^s}ds,$$

where $F(s) = \int_0^\infty f(x)x^{s-1}dx$ is a smooth function with $|F(\sigma + it)| \ll 1/|t|^{100}$.

(i) Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n} f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\zeta(s)Z(s)ds$$

where $Z(s)$ is the function appearing in Question 7.

(ii) Fix $\epsilon > 0$. Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n} f\Big(\frac{n}{y}\Big) = y\Big(\int_0^\infty f(x)dx\Big)\prod_p\Big(1 - \frac{2}{p^2} + \frac{1}{p^3}\Big) + O(y^{1/2+\epsilon})$$

(Compare the answer here to that in Question 7)

Solution 8. (i) We choose $\sigma = 2$ and apply the Mellin inversion statement to get

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n} f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\mu(n)^2\phi(n)}{n}\int_{2-i\infty}^{2+i\infty} F(s)\frac{y^s}{n^s}ds$$

Since $\Re(s) \geq 2$ we have $|n^{-s}| \ll 1/n^2$ and $|F(s)| \ll 1/|s|^2$, so the above expression converges absolutely, and so we may swap the order of summation and integration. This gives

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n} f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\Big(\sum_{n=1}^{\infty}\frac{\mu(n)^2\phi(n)}{n^{1+s}}\Big)ds$$

Recalling that $\sum_n \mu(n)^2\phi(n)/n^{1+s} = \zeta(s)Z(s)$ from the previous question gives the result.

(ii) As before, we move the line of integration to $\Re(s) = 1/2+\epsilon$ using Cauchy's residue Theorem. The only pole is at $s = 1$.

$$\frac{1}{2\pi i}\Big(\int_{1/2+\epsilon+iT}^{1/2+\epsilon-iT} + \int_{1/2+\epsilon-iT}^{2-iT} + \int_{2-iT}^{2+iT} + \int_{2+iT}^{1/2+\epsilon+iT}\Big)y^s F(s)\zeta(s)Z(s)ds = yZ(1)F(1).$$

On the second and fourth integrals $\zeta(s) \ll T$, $y^s \ll y^2$, $Z(s) \ll 1$ and $F(s) \ll 1/T^{100}$ so the contribution is $O(x/T^{99})$. On the first integral

$|y^s| \ll y^{1/2+\epsilon}$, $|\zeta(s)| \ll |s|$, $|Z(s)| \ll 1$ and $|F(s)| \ll 1/|s|^{100}$, so the contribution is $O(y^{1/2+\epsilon})$. Thus, letting $T \to \infty$ we find

$$\frac{1}{23\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\zeta(s)Z(s)ds = yZ(1)F(1) + O(y^{1/2+\epsilon}).$$

Question 9 (Bonus Question)**.** Let $\sigma(n) = \sum_{d|n} d$ be the sum of divisors of n. Following the approach of Question 7 or Question 8, obtain an asymptotic formula for $\sum_{n<x}\mu(n)^2\sigma(n)$ or $\sum_n \mu(n)^2\sigma(n)f(n/y)$.

Solution 9. We just give the argument for $\sum_n \mu(n)^2\sigma(n)f(n/y)$; the argument for the other case is essentially the same.

Note that $\sigma(n) \le n\tau(n) = n^{1+o(1)}$. Thus, using Mellin inversion and swapping the order of summation and integration as in the previous question, we have that

$$\sum_{n=1}^{\infty}\mu(n)^2\sigma(n)f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} y^s F(s)\Big(\sum_{n=1}^{\infty}\frac{\mu(n)^2\sigma(n)}{n^s}\Big)ds.$$

We may swap the order of summation since $|\sigma(n)/n^s| \ll 1/n^{3/2}$ when $\Re(s) = 3$, so everything converges absolutely. We have that for $\Re(s) > 2$

$$G(s) = \sum_{n=1}^{\infty}\frac{\mu(n)^2\sigma(n)}{n^s} = \prod_p\Big(1 + \frac{p+1}{p^s}\Big)$$

$$\Big(1+\frac{p+1}{p^s}\Big)\Big(1 - \frac{1}{p^{s-1}}\Big) = 1 - \frac{1}{p^{2s-2}} + \frac{1}{p^s} - \frac{1}{p^{2s-1}},$$

so $G(s) = \zeta(s-1)Z(s)$ where

$$Z(s) = \prod_p\Big(1 - \frac{1}{p^{2s-2}} + \frac{1}{p^s} - \frac{1}{p^{2s-1}}\Big),$$

and since $|1 - p^{-2s+2} + p^{-s} - p^{-2s+1}| \le 1 + 3p^{-2\sigma+2}$ for $\sigma = \Re(s) < 2$, we have that

$$|Z(s)| \ll \exp\Big(\sum_p \log\Big(1 + 3p^{-2\sigma+2}\Big)\Big) \ll \exp\Big(\sum_p O(p^{-2\sigma+2})\Big),$$

so $Z(s)$ converges absolutely for $\Re(s) > 3/2$. Thus we want to estimate

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} y^s F(s)Z(s)\zeta(s-1)ds.$$

We move the line of integration to $\Re(s) = 3/2 + \epsilon$. The only pole of the integrand is from $\zeta(s-1)$ at $s = 2$. by Cauchy's residue theorem

$$\frac{1}{2\pi i}\Big(\int_{3/2+\epsilon+iT}^{3/2+\epsilon-iT} + \int_{3/2+\epsilon-iT}^{3-iT} + \int_{3-iT}^{3+iT} + \int_{3+iT}^{3/2+\epsilon+iT}\Big)y^s F(s)\zeta(s-1)Z(s)ds = y^2 Z(2)F(2).$$

On the second and fourth contours, $|y^s| \ll y^3$, $|\zeta(s-1)| \ll T$, $|Z(s)| \ll 1$ and $|F(s)| \ll T^{-100}$, so these contribute $O(y^3T^{-99})$. On the first contour $|y^s| \ll y^{3/2}$, $|\zeta(s-1)| \ll (1+|s|)$, $|Z(s)| \ll 1$ and $|F(s)| \ll (1+|s|)^{-100}$, so this contributes $O(y^{3/2})$. Letting $T \to \infty$ then gives

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} y^s F(s)\zeta(s-1)Z(s)ds = y^2F(2)Z(2) + O(y^{3/2}).$$

This gives the desired asymptotic formula.

Question 10 (Bonus Question)**.** Let $\Psi(x,y) = \{n \le x : p|n \Rightarrow p \le y\}$ be the set of integers up to x which only involve prime factors of size at most y.

(i) Let $\alpha \in (0,1)$ be fixed. Show that as $x \to \infty$

$$\sum_{x^\alpha \le p \le x} \frac{1}{p} = \log\frac{1}{\alpha} + o(1).$$

(ii) Show that for $x^{1/2} \le y \le x$ we have

$$\Psi(x,y) = \Big(1 - \log\Big(\frac{\log x}{\log y}\Big) + o(1)\Big)x.$$

(iii) Show that for any $x \ge 1$ and $z \ge y > 0$

$$\Psi(x,y) = \Psi(x,z) - \sum_{p<y\le z} \Psi\Big(\frac{x}{p}, p\Big).$$

(iv) Deduce that for $x^{1/3} \le y \le x^{1/2}$

$$\Psi(x,y) = \Big(1 - \log 2 - \int_2^{\log x/\log y} \frac{1}{v}\Big(1 - \log(v-1)\Big)dv + o(1)\Big)x$$

(v) Define a function $\rho : [0,\infty) \to \mathbb{R}$ by $\rho(u) = 1$ if $u \le 1$ and for $u > 1$

$$\rho(u) = 1 - \int_1^u \rho(t-1)\frac{dt}{t}.$$

Show that parts *(ii)* and *(iv)* imply that for $u \le 3$

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x.$$

(vi) Show by induction that if

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x$$

for $u \le m$, then the same equation holds for $u \le m+1$. Deduce that for any fixed $u > 0$ we have

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x.$$

Solution 10. (i) This follows from partial summation and the prime number theorem. Actually it only requires Mertens' Theorem: we have $\sum_{p<x} 1/p = \log\log x - C + o(1)$, so $\sum_{x^\alpha<p<x} 1/p = \log\log x - \log\log x^\alpha + o(1) = \log 1/\alpha + o(1)$.

(ii) A number has at most one prime factor bigger than $y > x^{1/2}$, so by inclusion-exclusion

$$\Psi(x,y) = \#\{n \le x\} - \sum_{y<p\le x} \#\{n \le x : p|n\} = x + O(1) - \sum_{y<p\le x}\Big(\frac{x}{p} + O(1)\Big)$$
$$= x\Big(1 + o(1) - \sum_{y<p\le x}\frac{1}{p}\Big) = x\Big(1 - \frac{\log x}{\log y} + o(1)\Big).$$

(iii) Again this is inclusion-exclusion based on the largest prime factor $P^+(n)$ of n.

$$\Psi(x,y) = \sum_{\substack{n\le x\\ P^+(n)\le y}} 1 = \sum_{\substack{n\le x\\ P^+(n)\le z}} 1 - \sum_{\substack{n\le x\\ y<P^+(n)\le z}} 1 = \Psi(x,z) - \sum_{y<p\le z} \#\{n \le x : P^+(n) = p\}.$$

But, since the last set contains elements which are a multiple of p, we see that $\#\{n \le x : P^+(n) = p\} = \#\{m \le x/p : P^(m) \le p\} = \Psi(x/p,p)$.

(iv) Let $z = x^{1/2}$, and apply the above identity. We have

$$\Psi(x,y) = \Psi(x,x^{1/2}) - \sum_{y\le p\le x^{1/2}} \Psi\Big(\frac{x}{p},p\Big).$$

We now see that since $y \ge x^{1/3}$ we have $p \ge (x/p)^{1/2}$ for all p in the above sum, so we may apply (*ii*). This gives

$$\Psi(x,y) = \Big(1 - \log 2 + o(1)\Big)x - \sum_{y\le p\le x^{1/2}} \Big(1 - \log\Big(\frac{\log x/p}{\log p}\Big) + o(1)\Big)\frac{x}{p}.$$

By partial summation

$$\sum_{y\le p\le x^{1/2}} \Big(1 - \log\Big(\frac{\log x/p}{p}\Big)\Big)\frac{x}{p} = x\int_y^{x^{1/2}} \frac{1}{t\log t}\Big(1 - \log\Big(\frac{\log x/t}{\log t}\Big)\Big)dt + o(x)$$
$$= x\int_2^{\log x/\log y} \frac{1}{v}\Big(1 - \log(v-1)\Big)dv.$$

(We made a change of variables $v = \log x/\log p$ in the last line). This gives the result.

(v) Since $\rho(1) = 1$ for $u \le 1$, the result is trivial for $u \le 1$. We then see $\rho(u) = 1 - \int_1^u dt/t = 1 - \log u$ for $1 \le u \le 2$, so this agrees with (*ii*). Finally, for $2 \le u \le 3$ we see that

$$\rho(u) = 1 - \int_1^u \frac{\rho(t-1)}{t}dt = \rho(2) - \int_2^u \frac{\rho(t-1)dt}{t} = \rho(2) - \int_2^u \Big(1 - \log(t-1)\Big)\frac{dt}{t},$$

so this agrees with (iv).

(vi) Assume that for $u \leq m$ we have $\Psi(x, x^{1/u}) = (\rho(u) + o(1))x$. Then by (iii),

$$\Psi(x, x^{1/(u+1)}) = \Psi(x, x^{1/u}) - \sum_{x^{1/(u+1)} \leq p \leq x^{1/u}} \Psi\Big(\frac{x}{p}, p\Big).$$

But if $p \geq x^{1/(u+1)}$, we have $p^u \geq x/p$, so if $u \leq m$ we have

$$\Psi(x/p, p) = \Big(\rho\Big(\frac{\log x/p}{\log p} + o(1)\Big)\frac{x}{p}.$$

Thus

$$\begin{aligned}\Psi(x, x^{1/(u+1)}) &= \Big(\rho(u) + o(1)\Big)x - \sum_{x^{1/(u+1)} \leq p \leq x^{1/u}} \Big(\rho\Big(\frac{\log x/p}{\log p} + o(1)\Big)\frac{x}{p} \\ &= x\Big(\rho(u) + o(1) - \int_{x^{1/(u+1)}}^{x^{1/u}} \frac{1}{t \log t}\rho\Big(\frac{\log x/t}{\log t}\Big)dt\Big) \\ &= x\Big(\rho(u) + o(1) - \int_u^{u+1} \frac{1}{v}\rho(v-1)dv\Big).\end{aligned}$$

Here we substituted $v = \log x / \log t$ in the final line. But from the definition of ρ, for $u \geq 2$

$$\rho(u+1) = 1 - \int_1^{u+1} \frac{\rho(t-1)}{t}dt = \rho(u) - \int_u^{u+1} \frac{\rho(v-1)}{v}dv,$$

so the above expression is $x(\rho(u+1) + o(1))$, as required. Therefore we see that if the formula holds for $u \leq m$, it also holds for $u \leq m+1$.

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