C3.8 Analytic Number Theory, Michaelmas 2018 **Sheet 0**

The purpose of this sheet is to remind you of some basic techniques from analysis that we will use throughout the course.

Recall that $O(g(x))$ denotes some function $h(x)$ such that $|h(x)| \leq Cg(x)$ for some constant $C > 0$. This is very useful since we often won't care about the value of the constants in our bounds.

We will frequently need to compare sums to integrals, such as in the following two questions.

Question 1. Show that $\sum_{n=1}^{X} \frac{1}{n} = \log X + O(1)$ for $X \geq 1$.

Question 2. Show that $\sum_{n>X} \frac{1}{n^2} = O(\frac{1}{X})$ for $X \geq 1$.

We will frequently need to get approximations of one function by another function, as in the following two examples.

Question 3. Show that $(1+x)^3 = 1 + O(x)$ for $0 \leq x \leq 1$.

Question 4. Show that $\log(1-x) = -x + O(x^2)$ for $0 \leq x \leq \frac{1}{2}$.

We will often encounter infinite sums or products, and we need to know they converge to make sense of them

Question 5. Let $t \in \mathbb{R}$. Show that $\sum_{n=1}^{\infty} \frac{\log n}{n^{2+it}}$ converges absolutely.

Question 6. Show that $\prod_{n=1}^{\infty}(1 + \frac{1}{n^2})$ converges.

It will be important to get quick estimates of the size of infinite sums and integrals.

Question 7. Show that $\int_2^X \frac{dt}{t \log t} = O(\log\log X)$ for $X \geq 3$

Question 8. Show that $\sum_{n=2}^{X} \frac{1}{\log n} = O(\frac{X}{\log X})$ for $X \geq 3$.

We will repeatedly use the fact that limits of nice complex-valued functions remain nice if the convergence in the limit is uniform.

Question 9. Suppose that $f_1(z), f_2(z), \dots$ are a sequence of holomorphic functions on some domain $\mathcal{D}$ (i.e. open connected set), and that $f_j(z) \to f(z)$ as $j \to \infty$ uniformly for all $z \in \mathcal{D}$. Show that $f(z)$ is holomorphic on $\mathcal{D}$.

(Hint: Recall Morera's theorem that if $\int_\gamma f(z) = 0$ for all closed curves gamma on the domain on which f is defined, then f is holomorphic.)

Question 10. Show that $f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is a holomorphic function of s on $\Re(s) \geq 2$.

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Sketch Solutions

Solution 1. Since $1/t$ is decreasing, we have that for $n \geq 2$

$$\int_n^{n+1} \frac{dt}{t} \leq \frac{1}{n} \leq \int_{n-1}^{n} \frac{dt}{t}.$$

Therefore, summing the first inequality over $1 \leq n \leq X-1$ and adding back in the final term, and summing the second inequality over $2 \leq n \leq X$ and adding back in the first term gives

$$\log X + \frac{1}{X} = \int_1^X \frac{dt}{t} + \frac{1}{X} \leq \sum_{n=1}^{X} \frac{1}{n} \leq 1 + \int_1^X \frac{dt}{t} = 1 + \log X.$$

Rearranging, this gives $|\sum_{n\leq X} \frac{1}{n} - \log X| \leq 1$, as required.

Solution 2. Using the same idea as in the previous question we see

$$\sum_{n>X} \frac{1}{n^2} \leq \sum_{n>X} \int_n^{n+1} \frac{dt}{t^2} \leq \int_X^\infty \frac{dt}{t^2} = \frac{1}{X}.$$

Solution 3. For $0 \leq x \leq 1$ we see that

$$(1+x)^3 - 1 = 3x + 3x^2 + x^3 \leq 7x.$$

Thus $(1+x)^3 = 1 + O(x)$.

Solution 4. We have

$$\log(1-x) = -\int_{1-x}^{1} \frac{dt}{t}.$$

The integral is over an interval of length x, and the integrand is always between $1/(1-x)$ and 1. Therefore, for $0 \leq x \leq 1/2$

$$|\log(1-x) + x| \leq \frac{x}{1-x} - x = \frac{x^2}{1-x} \leq 2x^2$$

Solution 5. We have that

$$\sum_{n=M}^{\infty} \Big|\frac{\log n}{n^{2+it}}\Big| = \sum_{n=M}^{\infty} \frac{\log n}{n^2} \leq \int_{M-1}^{\infty} \frac{\log t}{t^2} dt.$$

To bound the integral, we note that

$$\log t = \int_1^t \frac{dx}{x} \leq \int_{t^{1/2}}^{t} \frac{dx}{t^{1/2}} + \int_1^{t^{1/2}} dx \leq 2t^{1/2}.$$

Therefore

$$\int_{M-1}^{\infty} \frac{\log t}{t^2} dt \leq \int_{M-1}^{\infty} \frac{2dt}{t^{3/2}} \leq \frac{4}{(M-1)^{1/2}},$$

which tends to zero as $M \to \infty$, as required.

Solution 6. Note that

$$\prod_{n=1}^{N}\Big(1+\frac{1}{n^2}\Big) = \exp(\sum_{n=1}^{N}\log\Big(1+\frac{1}{n^2}\Big)),$$

so the product converges if and only if the sum inside the exponential converges. Since $\log(1+x) = \int_1^{1+x}\frac{dt}{t} \le x$, we see that

$$\sum_{n=1}^{N}\log\Big(1+\frac{1}{n^2}\Big) \le \sum_{n=1}^{N}\frac{1}{n^2} \le 1+\int_1^N\frac{dt}{t^2} \le 2,$$

so they do both converge.

Solution 7. Substituting $u = \log t$ gives

$$\int_2^X \frac{dt}{t\log t} = \int_{\log 2}^{\log X}\frac{du}{u} = \log\log X - \log\log 2 \le \log\log X.$$

Solution 8.

$$\sum_{n=2}^{X}\frac{1}{\log n} \le \sum_{n=2}^{X^{1/2}} 1 + \sum_{n=X^{1/2}}^{X}\frac{2}{\log X} \le X^{1/2} + \frac{X}{\log X} \le \frac{3X}{\log X}.$$

Solution 9. Since $f_j(z) \to f(z)$ uniformly, for any path γ in $\mathcal{D}$ we have

$$\lim_{j\to\infty}\int_\gamma f_j(z) = \int_\gamma f(z).$$

But if γ is a closed curve, then $\int_\gamma f_j(z) = 0$ for all j since f_j is holomorphic. So $\int_\gamma f(z) = 0$ for all closed curves γ. By Morera's theorem, this means $f(z)$ is holomorphic.

Solution 10. Since $1/n^s$ is holomorphic, we just need to check that $\sum_{n=1}^{\infty} n^{-s}$ converges uniformly on $\Re(s) \ge 2$. But this is simple since

$$\sum_{n>N}\Big|\frac{1}{n^s}\Big| \le \sum_{n>N}\frac{1}{n^2} \le \frac{2}{N}.$$