C3.8 Analytic Number Theory, Michaelmas 2018 **Exercises 2**

Question 1. Give a simple description of the function $\phi \star 1$.

Question 2. (i) Recall $\phi(n) = \#\{d \leq n : \gcd(d,n) = 1\}$. Show that $\sum_n \phi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}$ for $\Re s > 2$.

(ii) Let $\sigma(n) = \sum_{d|n} d$. Show that $\sum_n \sigma(n) n^{-s} = \zeta(s)\zeta(s-1)$ for $\Re s > 2$.

(iii) Let $\lambda(n)$ be the Liouville function, the completely multiplicative function equal to -1 on the primes. Show that $\sum_n \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)}$ for $\Re s > 1$.

(iv) Let $\tau_k(n) = \sum_{d_1 \dots d_k = n} 1$ be the number of ways of writing n as a product of k integers. Show that $\sum_n \tau_k(n) n^{-s} = \zeta(s)^k$ for $\Re s > 1$.

Question 3. True or false: there is a constant C such that $\tau(n) \leqslant \log^C n$ for all sufficiently large n. (Do not assume the prime number theorem.)

Question 4. (i) Show that for all $Y \geq 1$

$$\frac{1}{Y} \sum_{n \geq 1} \Lambda(n) \lfloor \frac{Y}{n} \rfloor = \frac{1}{Y} \sum_{n \leqslant Y} \log n.$$

(ii) Show that

$$\sum_{n \leq Y} \log n = Y \log Y + O(Y).$$

(iii) Use part (i) and (ii) to show that

$$\sum_{X/2 < n \leqslant X} \Lambda(n) \ll X.$$

(*Hint:* $\lfloor 2x/d \rfloor - 2\lfloor x/d \rfloor \geq 0$, and is 1 when $x < d \leq 2x$.)

Question 5. Let $\Lambda_k(n) = (\mu \star \log^k)(n) = \sum_{de=n} \mu(d)(\log e)^k$. Show that

(i) $-\mu(n) \log n = \sum_{de=n} \mu(d)\Lambda(e)$.

(ii) $\Lambda_{k+1}(n) = \Lambda_k(n) \log n + (\Lambda_k \star \Lambda)(n)$.

(iii) $\Lambda_k(n) = 0$ unless n has at most k distinct prime factors.

(iv) If $n = p_1 \dots p_k$ (for distinct primes $p_1, \dots, p_k$) then $\Lambda_k(n) = k!(\log p_1) \dots (\log p_k)$.

(v) $0 \leq \Lambda_k(n) \leq (\log n)^k$.

(*Hint: Use (ii) in subsequent parts!*)

Question 6. Give an asymptotic for $\sum_{n \leqslant X} \phi(n)$.
(*Hint:* Using your answer to Question 1, or otherwise, first establish that the expression to be estimated is $\sum_{d \leqslant X} \mu(d) \sum_{m \leqslant X/d} m$.)

Question 7. (i) Show that

$$\tau(n) = \sum_{\substack{d|n \\ d\leq n^{1/2}}} 1 + \sum_{\substack{d|n \\ d<n^{1/2}}} 1.$$

(ii) Deduce that

$$\sum_{n\leq x}\tau(n) = 2\sum_{d\leq x^{1/2}}\frac{x}{d} - x + O(x^{1/2}).$$

(iii) Let γ be the constant

$$\gamma = \int_1^\infty \frac{\{t\}}{t(t-\{t\})}dt.$$

(Recall $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t.) Show that

$$\gamma = \sum_{1\leq d\leq x}\frac{1}{d} - \log x + O(1/x),$$

and deduce

$$\sum_{n\leq x}\tau(n) = x\log x + (2\gamma - 1)x + O(x^{1/2}).$$

Question 8. The aim of this question is to show that the prime number theorem implies that $M(X) = o(X)$, where $M(X) := \sum_{n\leqslant X}\mu(n)$.

(i) Prove that if $n \neq 1$ then

$$-\mu(n)\log n = \sum_{ab=n}\mu(a)(\Lambda(b) - 1).$$

(ii) Deduce that

$$|M(X)| \leqslant \frac{1}{\log X}\sum_a |\sum_{b\leqslant X/a}(\Lambda(b)-1)| + o(X).$$

(iii) Assuming the prime number theorem, show that indeed $M(X) = o(X)$.

james.maynard@maths.ox.ac.uk