Question 1. Let $q > 1$ be a positive integer. Let $e(x) = e^{2\pi i x}$, so $e(x/q)$ is a well-defined function for $x \in \mathbb{Z}/q\mathbb{Z}$. For any function $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, define the *discrete Fourier transform* by

$$\hat{f}(a) := \frac{1}{q} \sum_{b \in \mathbb{Z}/q\mathbb{Z}} f(b) e\Big(\frac{-ab}{q}\Big)$$

for $a \in \mathbb{Z}/q\mathbb{Z}$.

(i) For each integer a, show that

$$\sum_{b=1}^{q} e\Big(\frac{ab}{q}\Big) = \begin{cases} q, & \text{if } q \text{ divides } a, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Deduce the Fourier inversion formula:

$$f(a) = \sum_{b \in \mathbb{Z}/q\mathbb{Z}} \hat{f}(b) e\Big(\frac{ab}{q}\Big).$$

(iii) Show Parseval's formula:

$$\sum_{a \in \mathbb{Z}/q\mathbb{Z}} |f(a)|^2 = q \sum_{b \in \mathbb{Z}/q\mathbb{Z}} |\hat{f}(b)|^2.$$

Question 2. Define functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ by setting $F_1(x) = 1$ if $|x| \leqslant 1$, and 0 otherwise; and $F_2(x) = 1 - |x|$ if $|x| \leqslant 1$, and 0 otherwise.

(i) Calculate $\hat{F}_1(\xi)$ and $\hat{F}_2(\xi)$, showing that they make sense for all $\xi \in \mathbb{R}$.

(ii) Show that $\int_{-\infty}^{\infty} |\widehat{F}_1(\xi)| d\xi$ is infinite, but that $\int_{-\infty}^{\infty} |\widehat{F}_2(\xi)| d\xi$ is finite.

Question 3. Show that the function $\zeta'(s)/\zeta(s)$ has

(i) A simple pole at $s = \rho$, for every non-trivial zero ρ of $\zeta(s)$.

(ii) A simple pole at $s = -2, -4, \ldots$

(iii) A simple pole at $s = 1$.

(iv) No other poles in the complex plane.

Question 4. Recall from lectures that for $\Re(s) > -2$ we have

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - s(s+1)(s+2) \int_1^{\infty} \frac{(\{t\} - 3\{t\}^2 + 2\{t\}^3) dt}{12 t^{s+3}}$$

(i) Show, using induction, that for each integer $k \geq 2$ we have

$$\zeta(s) = \frac{1}{s-1} + Q_k(s) + s(s+1)\dots(s+k)\int_1^\infty \frac{P_k(\{t\})}{t^{s+k+1}}$$

in the region $\Re(s) > -k$, for some polynomials Q_k, P_k with rational coefficients. Deduce that $\zeta(-(2n+1))$ is a rational number of all positive integers n.

(ii) Show that

$$\Gamma(1/2) = \pi^{1/2}.$$

Deduce from this that $\Gamma(1/2+n)$ is a rational multiple of $\pi^{1/2}$ for all integers n.

(iii) Deduce that $\zeta(2n)$ is a rational multiple of π^{2n} for all positive integers n.

Question 5. Let $f, g \in \mathcal{S}(\mathbb{R})$, the set of Schwarz functions on $\mathbb{R}$.

(i) Show that

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx.$$

(ii) Let $r_t(x) = f(tx)$ and $s_t(x) = f(x+t)$. Show that

$$\hat{r_t}(\xi) = \frac{\hat{f}(\xi/t)}{t},$$
$$\hat{s_t}(\xi) = e^{2\pi i\xi t}\hat{f}(\xi).$$

(iii) Let $h_t(x) = e^{-\pi x^2/t^2}$. Show that as $t \to \infty$

$$\int_{-\infty}^{\infty} \hat{f}(x)h_t(x)dx \to \int_{-\infty}^{\infty} \hat{f}(x)dx$$

and

$$\int_{-\infty}^{\infty} f(x)\hat{h}_t(x)dx \to f(0).$$

(iv) By applying (iii) to $s_t(x)$, deduce the Fourier inversion formula for $\mathcal{S}(\mathbb{R})$:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}d\xi$$

(v) Deduce Plancherel's formula

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx.$$

Question 6. Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function such that $f(x)$ is non-zero only when $x \in [\alpha, \beta]$ for some finite $0 < \alpha < \beta$. Define the *Mellin Transform* $F : \mathbb{C} \to \mathbb{C}$ by

$$F(s) := \int_0^\infty f(x)x^{s-1}dx.$$

(i) Show that $g_\sigma(x) = f(e^x)e^{\sigma x}$ is a function in $\mathcal{S}(\mathbb{R})$ and that $F(\sigma + it) = \hat{g}_\sigma(-t/2\pi)$.

(ii) Using the Fourier inversion formula (Question 5 part (iv)), deduce the *Mellin Inversion Formula*: For any $\sigma \in \mathbb{R}$

$$f(x) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)x^{-s}ds.$$

(iii) Let $h_y(x) = f(x/y)$. Show the Mellin transform satisfies $H_y(s) = y^s F(s)$. Deduce that

$$\sum_{n=1}^\infty f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\zeta(s)ds.$$

(iv) It is a fact that $\zeta(s) \ll |s|^{101}$ for $\Re(s) > -100$ and $|s-1| \geq 1$ (This follows from the formula in Question 4 (i))

Using this and the Cauchy residue theorem, deduce that

$$\sum_{n=1}^\infty f\Big(\frac{n}{y}\Big) = y\int_0^\infty f(x)dx + O(y^{-100}).$$

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