C3.8 Analytic Number Theory, Michaelmas 2018 **Exercises 4**

Question 1. Prove that $\pi(X) \leqslant 2\pi(X/2)$ for X sufficiently large.

Question 2. Let p_n denote the n^{th} prime. Prove that

$$p_n = n \log n + n \log \log n + O(n).$$

Question 3. (i) Let $\theta \in (0,1)$ be such that $\Re(\rho) \leq \theta$ for all non-trivial zeros ρ. Deduce that for all $x \geq 2$

$$\sum_{n<x} \Lambda(n) = x + O(x^\theta (\log x)^2).$$

(ii) Let $\gamma \in (0,1)$ be such that for all $x \geq 2$

$$\sum_{n<x} \Lambda(n) = x + O(x^\gamma).$$

Show that $\Re(\rho) \leq \gamma$ for all zeros of $\zeta(s)$.
(Hint: Use partial summation to prove analytic continuation of ζ'/ζ)

(iii) Let $\alpha \in (0,1)$ be fixed. Show that if for all $x \geq 2$ we have

$$\sum_{n<x} \Lambda(n) = x + O\Big(x^\alpha \exp(\sqrt{\log x})\Big)$$

then in fact

$$\sum_{n<x} \Lambda(n) = x + O(x^\alpha (\log x)^2).$$

Question 4. (i) Show that for $\Re(s) > 1$ we have

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

(ii) Show that $3 + 4\cos(\theta) + \cos(2\theta) \geq 0$.

(iii) Using (i) and (ii), show that for $\sigma > 1$

$$3 \log \zeta(\sigma) + 4\Re \log \zeta(\sigma + it) + \Re\text{log}(\zeta(\sigma + 2it) \geq 0.$$

Deduce from this that for $\sigma > 1$

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$

(iv) Deduce from the above inequality that $\zeta(1 + it) \neq 0$.
(Hint: Consider $\sigma \to 1$)

Question 5. It is a fact that

$$\sum_{n<x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}).$$

where the sum is understood to be the limit as $T \to \infty$ of $\sum_{|\Im(\rho)|\le T} x^\rho/\rho$ over non-trivial zeros ρ (it is not absolutely convergent).

(i) Using this fact, show that $\zeta(s)$ must have at least one non-trivial zero.

(ii) Show that if ρ is a non-trivial zero of $\zeta(s)$, then so is $1-\rho$.

(iii) Let $\epsilon > 0$. Using Question 3, deduce that we cannot have for all $x \ge 2$ that

$$\sum_{n<x} \Lambda(n) = x + O(x^{1/2-\epsilon}).$$

Question 6. Recall from Sheet 3 Q6: If $f : \mathbb{R} \to \mathbb{C}$ is smooth and non-zero only on some interval $[\alpha, \beta] \subseteq (0, \infty)$ then for any $\sigma \in \mathbb{R}$

$$f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\frac{y^s}{n^s}ds,$$

where $F(s) = \int_0^\infty f(x)x^{s-1}dx$ is a smooth function with $|F(\sigma+it)| \ll 1/|t|^{100}$ and with no zeros or poles.

(i) Show that

$$\sum_{n=1}^{\infty} \Lambda(n) f\Big(\frac{n}{y}\Big) = \frac{-1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\frac{\zeta'(s)}{\zeta(s)}ds.$$

(ii) Deduce that

$$\sum_{n=1}^{\infty} \Lambda(n) f\Big(\frac{n}{y}\Big) = y\int_0^\infty f(t)dt - \sum_{\rho} y^\rho F(\rho) + O(y^{-1/4})$$

where $\sum_\rho$ is a sum over all non-trivial zeros of $\zeta(s)$ with multiplicity.

Question 7. For this question you may use the following fact: $|\zeta(\sigma+it)| \ll |t|^{1-\sigma}\log|t|$ for $\sigma \le 1$ and $|t| \ge 1$.

(i) Show using Perron's formula that for $2 \le T \le 2x$

$$\sum_{n<x} \frac{\mu(n)^2\phi(n)}{n} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\zeta(s)Z(s)ds + O\Big(\frac{x(\log x)^3}{T}\Big),$$

where $c = 1 + 1/\log x$ and for $\Re(s) > 1$

$$Z(s) = \prod_p \Big(1 - \frac{1}{p^{2s}} - \frac{1}{p^{s+1}} + \frac{1}{p^{2s+1}}\Big).$$

(ii) Show that the product of for $Z(s)$ converges absolutely for $\Re(s) > 1/2$.

(iii) Let $\epsilon = 1/1000$. By moving the line of integration to $\Re(s) = 1/2+\epsilon$, show that

$$\sum_{n<x} \frac{\mu(n)^2\phi(n)}{n} = x\prod_p\Big(1-\frac{2}{p^2}+\frac{1}{p^3}\Big)+O\Big(x^{1/2+\epsilon}T^{1/2-\epsilon}\log x\Big)+O\Big(\frac{x(\log x)^3}{T}\Big)$$

(iv) Deduce that

$$\sum_{n<x} \frac{\mu(n)^2\phi(n)}{n} = x\prod_p\Big(1-\frac{2}{p^2}+\frac{1}{p^3}\Big)+O\Big(x^{2/3+\epsilon}\Big).$$

Question 8. Recall from Sheet 3 Q6: If $f:\mathbb{R}\to\mathbb{C}$ is smooth and non-zero only on some interval $[\alpha,\beta]\subseteq(0,\infty)$ then for any $\sigma\in\mathbb{R}$

$$f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\frac{y^s}{n^s}ds,$$

where $F(s) = \int_0^\infty f(x)x^{s-1}dx$ is a smooth function with $|F(\sigma+it)| \ll 1/|t|^{100}$.

(i) Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n} f\Big(\frac{n}{y}\Big) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s F(s)\zeta(s)Z(s)ds$$

where $Z(s)$ is the function appearing in Question 7.

(ii) Fix $\epsilon > 0$. Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2\phi(n)}{n} f\Big(\frac{n}{y}\Big) = y\Big(\int_0^\infty f(x)dx\Big)\prod_p\Big(1-\frac{2}{p^2}+\frac{1}{p^3}\Big)+O(y^{1/2+\epsilon})$$

(Compare the answer here to that in Question 7)

Question 9 (Bonus Question)**.** Let $\sigma(n) = \sum_{d|n} d$ be the sum of divisors of n. Following the approach of Question 7 or Question 8, obtain an asymptotic formula for $\sum_{n<x}\mu(n)^2\sigma(n)$ or $\sum_n \mu(n)^2\sigma(n)f(n/y)$.

Question 10 (Bonus Question)**.** Let $\Psi(x,y) = \{n\le x : p|n \Rightarrow p\le y\}$ be the set of integers up to x which only involve prime factors of size at most y.

(i) Let $\alpha\in(0,1)$ be fixed. Show that as $x\to\infty$

$$\sum_{x^\alpha\le p\le x}\frac{1}{p} = \log\frac{1}{\alpha}+o(1).$$

(ii) Show that for $x^{1/2} \leq y \leq x$ we have

$$\Psi(x,y) = \Big(1 - \log\Big(\frac{\log x}{\log y}\Big) + o(1)\Big)x.$$

(iii) Show that for any $x \geq 1$ and $z \geq y > 0$

$$\Psi(x,y) = \Psi(x,z) - \sum_{p<y\leq z} \Psi\Big(\frac{x}{p}, p\Big).$$

(iv) Deduce that for $x^{1/3} \leq y \leq x^{1/2}$

$$\Psi(x,y) = \Big(1 - \log 2 - \int_2^{\log x/\log y} \frac{1}{v}\Big(1 - \log(v-1)\Big)dv + o(1)\Big)x$$

(v) Define a function $\rho : [0,\infty) \to \mathbb{R}$ by $\rho(u) = 1$ if $u \leq 1$ and for $u > 1$

$$\rho(u) = 1 - \int_1^u \rho(t-1)\frac{dt}{t}.$$

Show that parts *(ii)* and *(iv)* imply that for $u \leq 3$

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x.$$

(vi) Show by induction that if

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x$$

for $u \leq m$, then the same equation holds for $u \leq m+1$. Deduce that for any fixed $u > 0$ we have

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x.$$

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