Algebraic Geometry

Elizabeth Baldwin, Gergely Bérczi

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1 Introduction

Algebraic geometry is the study of algebraic varieties: an algebraic variety is, roughly speaking, a locus defined by polynomial equations. The well-known parabola, given as the graph of the function $f(x) = x^2$, is an immediate example: it is the zero locus of the polynomial $y - x^2$ in $\mathbb{R}^2$.
One of the advantages of algebraic geometry is that it is purely algebraically defined and applies to any field, including fields of finite characteristic. It is geometry based on algebra rather than on calculus, but over the real or complex numbers it provides a rich source of examples and inspiration to other areas of geometry.
A short historical background:

- Some of the roots of algebraic geometry date back to the work of the Hellenistic Greeks from the 5th century BC.
- The birthplace of modern algebraic geometry is Italy in the 19th century. The Italian school studied systematically projective varieties, and gave geometric proofs. Many objects got their names from Italian mathematicians. (Cremona, Castelnuovo, del Pezzo, Segre, Veronese, Severi, etc.)
- In the beginning of the 20th century the language and techniques found their limit and a stronger framework was needed. O. Zariski gave algebraic geoemtry a firm algebraic foundation and A. Weil studied varieties as abstract algebraic objects, not as subsets of affine or projective spaces.
- Serre in the '50s developed sheaf theory, and was a founder of the French algebraic geometry school.
- Grothendieck in the '60s introduced a much larger class of objects called schemes. These are varieties with multiple components and Serre's sheaf theory is crucial to develop this notion.
- Deligne and Mumford in the late '60s introduced stacks. These are more abstract objects appearing for example in the situation when we study quotients of varieties with group actions. Stacks have become and are becoming popular in the study of certain moduli spaces.

Throughout these notes, k is an algebraically closed field and any ring will be commutative
Background material: This course heavily builds on some basic results in commutative algebra. It is very much recommended to visit the Part C Commutative Algebra course, or study the notes of the course or the relevant books in the following bibliography.

1. K. A. Smith et al: An Invitation to Algebraic Geometry
2. Miles Reid: Undergraduate Algebraic Geometry
3. Miles Reid: Undergraduate Commutative Algebra
4. Atiyah-Macdonald: Introduction to Commutative Algebra
5. R. Hartshorne: Algebraic Geometry, Chapter I (more advanced)
6. Eisenbud-Harris: The Geometry of Schemes

2 Affine Algebraic Varieties

2.1 Affine varieties

Let k be an algebraically closed field, and k^n the n-dimensional vector space over k.

Definition 2.1. A subset $X \subseteq k^n$ is a variety if there is an ideal $I \subseteq k[x_1, \ldots, x_n]$ such that

$$X = \mathbb{V}(I) = \{x \in k^n | f(x) = 0,\ \forall\ f \in I\}$$

We say that X is the vanishing set of I.

Recall that $R = k[x_1, \ldots, x_n]$ is a Noetherian ring, that is, all its ideals are finitely generated. If $I = (f_1, \ldots, f_r)$ is generated by r elements then

$$\mathbb{V}(I) = \mathbb{V}(f_1, \ldots, f_r) = \{x \in k^n | f_i(x) = 0, i = 1, \ldots, r\}.$$

Example 2.2.

1. $\mathbb{V}(x_1, x_2) \subseteq k^2$ is $\{0\}$, the origin.
2. $\mathbb{V}(x_1, x_2) \subseteq k^3$ is $\{(0, 0, x_3) : x_3 \in k\}$, the third coordinate axis.
3. $\mathbb{V}(x_1 - a_1, \ldots, x_n - a_n) = \{(a_1, \ldots, a_n)\} \in k^n$
4. $\mathbb{V}(1) = \emptyset$, where 1 denotes the constant 1 polynomial.
5. For $f \in k[x_1, x_2]$, $\mathbb{V}(f) \subset k^2$ is an affine plane curve. If the degree of f is 2 this is called *conic*, if the degree is 3 it is a *cubic curve.*
6. $\mathbb{V}(f) \subseteq k^n$ for a polynomial f is called a *hypersurface.*
7. $\mathbb{V}(f)$ where $f = a_1x_1 + \ldots + a_nx_n$ is a linear form is called a *hyperplane.*
8. Let k^{n^2} be the vector space of $n \times n$ matrices over k, then the determinant det is a polynomial in the entries, and $SL(n, k) = \mathbb{V}(\det -1)$.

9. A **non-example**: open balls in $\mathbb{C}^n$. Say, $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ is not a variety.

Definition 2.3. The Zariski topology on k^n is the topology whose closed sets are the affine varieties $\mathbb{V}(I)$ for $I \subset k[x_1, \ldots, x_n]$.

We use the notation $\mathbb{A}^n_k$ for "k^n with the Zariski topology". We leave as an exercise on Sheet 1 to show that this is indeed a well-defined topology, that is, union of opens is open and intersection of finitely many closed is closed in the Zariski topology. (What is the ideal which defines the union and intersection of two varieties?)
We collect the main features of our new topology versus the well-known Euclidean topology in the following table:

Euclidean Topology	Zariski Topology
Analytically defined	Algebraically defined
Metric spaces	No notion of distance if $k \neq \mathbb{C}$
Works only for $\mathbb{C}^n$ or $\mathbb{R}^n$	Works for any (alg closed) k
$\mathbb{C}^n, \mathbb{R}^n$ is not compact	$\mathbb{A}^n_k$ is compact (arbitrary covers have finite subcovers)
Hausdorff	Not Hausdorff in general
Small open sets	Every open set is dense (*)

Definition 2.4. The Zariski topology on an affine variety $X \subseteq \mathbb{A}^n_k$ is the restricted topology, that is, the closed subsets of X are $\{X \cap \mathbb{V}(I) : I \subseteq k[x_1, \ldots, x_n]\}$.

Definition 2.5. An affine variety $X \subset \mathbb{A}^n_k$ is called reducible if it can be written as a non-trivial union of two subvarieties $X = X_1 \cup X_2$ where $X_1 \neq \emptyset, X_2 \neq \emptyset$. Otherwise it is called irreducible.

Example. For example, $\mathbb{V}(x_1x_2) \subseteq \mathbb{A}^2$ is reducible: $\mathbb{V}(x_1x_2) = \mathbb{V}(x_1) \cup \mathbb{V}(x_2)$ is the union of the coordinate axes.
Note that

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) | \mathrm{Im}(z_1) \geq 0\} \cup \{(z_1, \ldots, z_n) | \mathrm{Im}(z_1) \leq 1\}$$

is reducible in the Euclidean topology, but irreducible in the Zariski topology.
Property (*) does not hold for every affine variety, just for irreducible ones.

Definition 2.6. Let $X \subseteq \mathbb{A}^n$ be a subset. Define

$$\mathbb{I}(X) = \{f \in k[x_1, \ldots, x_n] | f(x) = 0 \ \forall x \in X\}$$

its *vanishing ideal* in $k[x_1, \ldots, x_n]$.

Example. Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Does $\mathbb{I}(\mathbb{V}(I)) = I$ hold? The answer is: not necessarily. Take for example $I = (x^2) \subset k[x, y]$. We have $\mathbb{V}(I) = \mathbb{V}(x^2) = \{(0, y) : y \in k\} \cong \mathbb{A}^1$. However, $\mathbb{I}(\mathbb{V}(I)) = (x)$.
Note that the ideals (x^2) and (x) define the same subvariety of $\mathbb{A}^2$. But not the same scheme – the ideal (x^2) corresponds to the y-axis with multiplicity 2! This is the fundamental idea of defining and working with schemes instead of varieties. The Nullstellensatz below will give the relationship between $\mathbb{I}(\mathbb{V}(I))$ and I in general.

2.2 Morphisms

Definition 2.7. A *morphism of affine spaces* $F : \mathbb{A}^n \to \mathbb{A}^m$ is given by a polynomial map $x = (x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_m(x))$, where $f_i \in k[x_1, \ldots, x_n]$.
A morphism $F : X \to Y$ of affine varieties $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ is given by a polynomial map $\mathbb{A}^n \to \mathbb{A}^m$ that restricts to X, such that $F(X) \subseteq Y$.
An *isomorphism* is a morphism which has an inverse morphism.

Remark. The image of a morphism need not be an affine variety. For example the image of the map $\mathbb{A}^2 \to \mathbb{A}^1, (x, y) \mapsto x$ restricted to the subvariety $\mathbb{V}(xy - 1) \subset \mathbb{A}^2$ is $\mathbb{A}^1 \setminus \{point\}$, which is not a closed subvariety in $\mathbb{A}^1$

3 Projective Varieties

3.1 Review of Projective Space

Let $k^* = k \setminus \{0\}$ be the group of units in k, i.e the nonzero elements. Ways to think of projective space:

Definition 3.1.
1. $\mathbb{P}^n_k = \mathbb{A}^{n+1} \setminus \{0\}/\sim$ where $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for $\lambda \in k^*$. A point of $\mathbb{P}^n$ is often denoted by $[x_0 : \ldots : x_n]$, where $x_0, \ldots, x_n$ are the *homogeneous coordinates.*
2. For $k = \mathbb{C}$, $\mathbb{P}^n_{\mathbb{C}} = S^n/\sim$, that is $\{x \in \mathbb{C}^{n+1} | \|x\| = 1\}$/"antipodal points". Note that $\mathbb{P}^1_{\mathbb{C}}$ is the Riemann sphere.
3. $\mathbb{P}^n_k$ as compactified affine space: $\mathbb{P}^n_k = \mathbb{A}^n \cup \mathbb{P}^{n-1} = \{[x_0 : \ldots : x_n] | x_0 \neq 0\} \cup \{[x_0 : \ldots : x_n] | x_0 = 0\}$ the n-dimensional affine space with a hyperplane at infinity. By induction, we have the standard decomposition $\mathbb{P}^n_k = \mathbb{A}^n_k \cup \mathbb{A}^{n-1}_k \ldots \cup \mathbb{A}^0_k$.

3.2 Projective Varieties

Recall that there are no non-constant holomorphic functions on $\mathbb{P}^1_k$=Riemann sphere, so zero loci of polynomials are trivial. Instead of this, we use homogeneous polynomials on k^{n+1}.

Definition. A polynomial $F \in k[x_0, \dots, x_n]$ is *homogeneous* of degree d if every monomial $x_0^{i_0} \dots x_n^{i_n}$ which appears with nonzero coefficient has degree $d = i_0 + \dots + i_n$. This is equivalent to saying $F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n)$ for all $\lambda \in k$.
A *homogeneous ideal* $I = (F_1, \dots, F_r) \subset k[x_0, \dots, x_n]$ is an ideal generated by homogeneous polynomials $F_1, \dots, F_r$ (whose degree are not necessarily the same)

Note that the value of F is not well defined at a point $[x_0 : \dots : x_n]$ of $\mathbb{P}^n$, but the zero locus is well-defined, since $F(x_0, \dots, x_n) = 0 \Leftrightarrow F(\lambda x_0, \dots \lambda x_n) = 0$.

Definition 3.2. $X \subset \mathbb{P}^n_k$ is a *projective variety* if it is the vanishing set of a homogeneous ideal $I = (F_1, \dots, F_r)$, that is

$$X = \mathbb{V}(I) = \{x \in \mathbb{P}^n_k | F_i(x) = 0 \ i = 1, \dots, r\}$$

Definition 3.3. The Zariski topology on $\mathbb{P}^n_k$ is the topology whose closed subsets are projective subvarieties. The Zariski topology on $X \subset \mathbb{P}^n_k$ is the induced topology.

Example.

1. If $L = a_0 x_0 + \dots + a_n x_n$ is a non-zero linear form then $\mathbb{V}(L) = \{[x_0 : \dots : x_n] | L(x_0, \dots, x_n) = 0\}$ is called a *projective hyperplane*. We denote by $H_i = \{[x_0 : \dots : x_n] | x_i = 0\}$ the ith *coordinate hyperplane*. Its complement is

   $$U_i = \{[x_0 : \dots : x_n] \in \mathbb{P}^n_k | x_i \neq 0\}$$

   and U_i is called the ith *coordinate chart*. There is a bijection map $\phi_i : U_i \to \mathbb{A}^n_k$ defined as

   $$[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right)$$

   where the element under hat is removed. The inverse is

   $$(x_1, \dots, x_n) \mapsto [x_1 : \dots : x_{i-1} : 1 : x_i : \dots x_n]$$

   We leave as an exercise that ϕ_i is a homeomorphism in the Zariski topology (see the first Problem sheet).

   If $X \subset \mathbb{P}^n$ is a projective variety then $X = \bigcup_{i=0}^n X \cap U_i$ is an open cover of X.

2. *Projective hypersurfaces* are defined as $\mathbb{V}(F) \subseteq \mathbb{P}^n$, where F is homogeneous.

3. Curves of the form $\mathbb{V}(y^2 z - x(x - z)(x - \lambda z)\}$ are called *elliptic curves*. (see the algebraic curves course)

Definition 3.4. The *affine cone* over a projective variety $X = \mathbb{V}(I) \subseteq \mathbb{P}^n_k$ is the affine variety $\hat{X} = \mathbb{V}(I) \subset \mathbb{A}^{n+1}_k$.

The map $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ such that $x \mapsto [x]$ restricts to $\hat{X} \setminus \{0\} \to X$. If $x \in \hat{X}$ then $\lambda x \in \hat{X}$ for all $\lambda \in k$, because $F(x) = 0$ implies that $F(\lambda x) = \lambda^d F(x) = 0$. So the affine cone is indeed a cone in $\mathbb{A}^n_k$:it is the union of lines represented by the points of X in $\mathbb{P}^n_k$.

Example 3.5. Consider $X = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$ and $\hat{X} = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$. We have $U_z = \{[x : y : 1] | x, y \in k\}$ and so $X \cap U_z = \mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$, which is the analogue of a circle. Now the points outside U_z are $H_z = \{[x : y : 0] : x, y \in k\}$ and $H_z \cap X = \mathbb{V}(x^2 + y^2 - 0)$ which includes points $[1 : i : 0], [1 : -i : 0]$. Can we see these points as limit points? Yes, set $\widetilde{y} = iy$ and the equation now is $x^2 - y^2 = 1$. The real part is a hyperbola, and it has asymptotes, which lines correspond to the points $[1 : i : 0], [1 : -i : 0]$.

Definition 3.6. If $X \subseteq \mathbb{A}^n \subset \mathbb{P}^n$ is an affine variety with a fixed embedding of $\mathbb{A}^n \subset \mathbb{P}^n$, then its *projective closure* $\overline{X} \subseteq \mathbb{P}^n$ is the smallest projective variety containing $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$.

How can we find the projective closure? The process is called homogenisation, and goes as follows. Let $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree d. Then $f = f_0 + \ldots + f_d$ where f_i is homogeneous of degree i and $f_d \neq 0$. The homogenization of f is the homogeneous polynomial

$$F = x_0^d f_0 + x_0^{d-1} f_1 + \ldots + x_0 f_{d-1} + f_d$$

If $X = \mathbb{V}(F) \subseteq \mathbb{P}^n$, then $X \cap U_0 = \mathbb{V}(f) \subseteq U_0 \simeq \mathbb{A}^n$.
We will not prove the following theorem, but see examples on the first problem sheet.

Theorem 3.7. *Let $X \subseteq \mathbb{A}^n$ be an affine variety, take $\mathbb{I}(X) \subseteq k[x_1, \ldots, x_n]$. Let $\widetilde{I}$ be the ideal generated by the homogenisation of all elements of $\mathbb{I}(X)$. Then $\overline{X} = \mathbb{V}(\widetilde{I}) \subseteq \mathbb{P}^n$ is the projective closure of $\overline{X} \cap U_0 = X$.*

Remark. We can have isomorphic affine varieties with non isomorphic projective closures: $\mathbb{V}(y - x^2), \mathbb{V}(y - x^3)$, are isomorphic subvarieties of $\mathbb{A}^2$, but their projective closure in $\mathbb{P}^2$ are not isomorphic, see the first problem sheet.

Example 3.8. $X = \mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2 = U_z \subset \mathbb{P}^2$ has projective closure $\overline{X} = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$

3.3 Morphisms of Projective Varieties

We start with an example.
The polynomial map $\phi : \mathbb{P}^1 \to \mathbb{P}^2, [s : t] \mapsto [s^2 : st : t^2]$ is well-defined, and restricts to a map $\mathbb{P}^1 \to \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$. This map is in fact a bijection, but its inverse is given as:

$$\begin{tikzcd} {[s^2 : st : t^2]} \arrow[r, "?"] & {[s : t]} \arrow[r, equal, "t \neq 0"] \arrow[d, equal, "s \neq 0"] & {[st : t^2]} \\ & {[s^2 : st]} & \end{tikzcd}$$

So define the map $\phi^{-1} : \mathbb{V}(xz - y^2) \to \mathbb{P}^1$ such that

$$[x : y : z] \to \begin{cases} [x : y] & \text{if } x \neq 0 \\ [y : z] & \text{if } z \neq 0 \end{cases}$$

On the overlap, ie when $x \neq 0, z \neq 0 (\Rightarrow y \neq 0)$, we have $[x : y] = [xz : yz] = [y^2 : yz] = [y : z]$. So this is a well defined map, and $\phi \circ \phi^{-1} = Id_{\mathbb{P}^1}$.
This suggests that morphism are locally polynomial maps: there is an open cover such that the map is polynomial on every patch.

Definition 3.9. Let $X \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ be projective varieties. A map $F : X \to Y$ is a *morphism of projective varieties* if for all $p \in X$ there exists an open neighborhood $p \in U \subseteq X$, such tha there exist homogeneous polynomials $f_0, \dots, f_m \in k[x_0, \dots, x_n]$ of the same degree such that $F|_U : U \to Y$ agrees with $[q] \to [f_0(q) : \dots : f_m(q)]$.
An *isomorphism of projective varieties* is a morphism with an inverse morphism.

Remark. Note that the f_i must be of the same degree in the above. Also that $f_i(p) \neq 0$ for at least one i for all $p \in U$. We will see later that the open cover in the definition of a morphism can always be chosen to be an affine cover, i.e U is an open subset of an affine variety.

Example. (Another example of a morphism) Projection from a point. Suppose that $X \subseteq \mathbb{P}^n$ and $p \in \mathbb{P}^n$ such that $p \notin X$. Fix $H = \mathbb{V}(L) \cong \mathbb{P}^{n-1}$ a hyperplane (L is a linear form) such that $p \notin H$. Define a map $\pi_p : X \to H \cong \mathbb{P}^{n-1}$ such that $q \to \overline{qp} \cap H$, where $\overline{qp}$ is the line passing through points p, q.
Pick coordinates on $\mathbb{P}^n$ so that $p = [1 : 0 : \dots : 0]$. $H = \{[x_0 : \dots : x_n] | x_0 = 0\}$ (such a choice can be easily made: in $\mathbb{A}^{n+1}$ choose basis starting with a representative for p, then vectors spanning H, $p \notin H$ says this can be chosen to be a basis). Then the projection π_p is given by $\pi_p : [x_0, \dots, x_n] \mapsto [0 : x_1 : \dots : x_n]$.

Definition 3.10. (Projective Equivalence) We say that $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^n$ are *projectively equivalent* if they are transformed into each other by a linear change of coordinates in $\mathbb{P}^n$. That is there exists a linear transformation $A \in GL(n+1, k)$ which esteblishes an isomorphism $A : X \to Y, [x_0 : \dots : x_n] \mapsto [Ax_0 : \dots : Ax_n]$ between X and Y. (The inverse is given then by the inverse matrix A^{-1}).

Remark. Note that in this situation A and λA define the same transformation in projective space for $\lambda \in k^*$, and in fact the group which acts is

$$PGL(n, k) = \mathbb{P}(GL(n+1, k)) = GL(n+1, k)/k^*$$

Remark. We will see in §5 that projective equivalence is a stronger relation than isomorphism, namely, the homogeneous coordinate ring of a projective variety depends on the embedding into the projective space, and therefore isomorphic projective varieties might have non-isomorphic homogeneous coordinate rings.
However, projectively equivalent varieties have isomorphic coordinate rings.
The projective hyperplanes $H_0, H_1 \subseteq \mathbb{P}^2$ are projectively equivalent via the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

On the other hand, $\mathbb{P}^1 \cong H_0 \subseteq \mathbb{P}^2$ and $\mathbb{P}^1 \cong \mathbb{V}(xz - y^2)$, but $\mathbb{V}(xz - y^2)$ is not projectively equivalent to H_1, because they have different degrees, see the proof later.

4 Pretty Examples

4.1 Veronese maps

The goal of this section is to construct different embeddings of the same projective variety. That is, for a given $X \subset \mathbb{P}^n$ construct other embeddings $Y \subset \mathbb{P}^m$ such that X and Y are isomorphic. As a first step the Veronese maps embed $\mathbb{P}^n$ non-trivially into a bigger projective space $\mathbb{P}^m$ $(m > n)$.

Definition 4.1. The degree d *Veronese map* $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ is defined as $[x_0 : \ldots : x_n] \mapsto [\ldots : x^I : \ldots]$, where the coordinates run over a basis of monomials of degree d in $x_0, \ldots, x_n$. Here $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ stand for multiindices with $i_0 + \ldots + i_n = d$ and $x^I = x_0^{i_0} \cdots x_n^{i_n}$.

Remark.

1. The dimension of the target projective space is $\sharp\{(i_0, \ldots, i_n) : i_0 + \ldots + i_n = d\} - 1 = \binom{n+d}{d} - 1$. ν_d depends on the order of the multiindeces, but by changing the order results projectively equivalent images.

2. An other interpretation of the Veronese map is the following:

$$\nu_d : \mathbb{P}(\mathbb{A}^{n+1}) \to \mathbb{P}(\mathrm{Sym}^d \mathbb{A}^{n+1})$$
$$[v] \mapsto [v^d],$$

where $\mathrm{Sym}^{\mathrm{d}} \mathbb{A}^{\mathrm{n+1}}$ is the d symmetric product of the vector space $\mathbb{A}^{n+1}$.

Example.

1. The image of the Veronese map $\nu_d : \mathbb{P}^1 \to \mathbb{P}^n$ is called the *rational normal curve* of degree d. For example $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is given by $[x_0 : x_1] \mapsto [x_0^2 : x_0x_1 : x_1^2]$. The image is the projective variety $\nu_2(\mathbb{P}^1) = \mathbb{V}(z_{(2,0)}z_{(0,2)} - z_{(1,1)}^2)$, where $z_{i,j}$ is the homogeneous coordinate on $\mathbb{P}^2$ where x_ix_j appears in the map.

2. The Veronese surface is the image of the degree 2 Veronese embedding in $\mathbb{P}^5$:

$$[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2]$$

Proposition 4.2. *The image of $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ is the projective variety*

$$W = \mathbb{V}(\{z_Iz_J - z_Kz_L : I, J, K, L \in \mathbb{N}^{n+1}, I + J = K + L\}),$$

and the image is isomorphic to $\mathbb{P}^n$. In particular, $\nu_d(\mathbb{P}^n)$ is an intersection of quadrics, the zero locus of quadratic polynomials. Here the sum of the multiindices $I = (i_0, \dots, i_n), J = (j_0, \dots, j_n)$ is $I + J = (i_0 + j_0, \dots, i_n + j_n)$.

Proof.
Recall that the homogeneous coordinates of $\mathbb{P}^{\binom{n+d}{d}-1}$ are indexed by the degree d monomials in $(n + 1)$ variables; we can write them as z_I for $I = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$ with $\sum i_j = d$.
If $I + J = K + L$, then $z_Iz_J - z_Kz_L$ vanishes on $\nu_d(\mathbb{P}^n)$ as

$$x^Ix^J - x^Kx^L = x^{I+J} - x^{K+L} = 0.$$

This implies that $\nu_d(\mathbb{P}^n) \subset W$. To show that $W \subset \nu_d(\mathbb{P}^n)$ and ν_d is an isomorphism onto W we construct the inverse of ν_d, that is a morphism $\phi : W \to \mathbb{P}^n$ such that $\phi \circ \nu_d = id_{\mathbb{P}^n}$ and $\nu_d \circ \phi = id_W$.
Let $z = [\dots : z_I : \dots] \in W$. Then one of the coordinates $z_{(d,0,\dots,)}, z_{(0,d,\dots)}, \dots, z_{(0,\dots,,0,d)}$ is nonzero, otherwise from the equations $z_Iz_J = z_Kz_L$ all of them would be zero, which cant occur for a point in $\mathbb{P}^m$. Indeed, assume that $z_{(d,0,\dots,0)} = z_{(0,d,\dots,0)} = \dots = z_{(0,\dots,,0,d)} = 0$ but $z_{(i_0,\dots,i_n)} \neq 0$. Without loss of generality we can assume that $i_0 > 0$, and it is maximal, that is, $z_{(j_0,\dots,j_n)} = 0$ for $j_0 > i_0$. Note that $i_0 < d$, so there is an other index, say i_1, such that $d > i_1 > 0$. The left hand side of the equation $z^2_{(i_0,\dots,i_n)} = z_{(i_0+1,i_1-1,\dots,i_n)}z_{(i_0-1,i_1+1,\dots,i_n)}$ is nonzero, therefore $z_{(i_0+1,i_1-1,\dots,i_n)} \neq 0$, which contradicts to the maximality of i_0.
Let $U_i \subset W$ be the subset of W where the coordinate indexed by x_i^d is nonzero. So the sets $U_0, \dots, U_n$ cover W and we can define a map

$$\begin{array}{rcl} \phi_i : U_i & \to & \mathbb{P}^n \\ z & \mapsto & [z_{(1,0,\dots,d-1^i,\dots,0)} : z_{(0,1,0,\dots,d-1^i,0,\dots,0)} : \dots : z_{(0,\dots,d-1^i,0,\dots,1)}] \end{array}$$

for $z \in U_i$. That is, we send z to the $(n+1)$-tuple of its coordinates indexed by $x_0x_i^{d-1}, \dots, x_nx_i^{d-1}$.
These maps agree on the overlaps $U_i \cap U_j$, where from the equation $z_{(0,\dots,1^a,\dots d-1^j,\dots,0)}z_{(0,\dots,d^i,\dots,0)} = z_{(0,\dots,1^a,\dots d-1^i,\dots,0)}z_{(0,\dots,1^i,\dots d-1^j,\dots,0)}$ we get

$$z_{(0,\dots,1^a,\dots,d-1^i,\dots,0)} = \frac{z_{(0,\dots,d^i,\dots,0)}}{z_{(0,\dots,1^i,\dots,d-1^j,\dots,0)}} z_{(0,\dots,1^a,\dots,d-1^j,\dots,0)}$$

and therefore

$$[z_{(1,0,\dots,d-1^i,\dots,0)} : z_{(0,1,0,\dots,d-1^i,0,\dots,0)} : \dots : z_{(0,\dots,d-1^i,0,\dots,1)}] =$$
$$[z_{(1,0,\dots,d-1^j,\dots,0)} : z_{(0,1,0,\dots,d-1^j,0,\dots,0)} : \dots : z_{(0,\dots,d-1^j,0,\dots,1)}]$$

So these maps patch together to define a morphism of projective varieties $\phi : W \to \mathbb{P}^n$. The composition $\phi \circ \nu_d : \mathbb{P}^n \to \nu_d(\mathbb{P}^n) \to \mathbb{P}^n$ is

$$[x_0 : \dots : x_n] \mapsto v_d(x) \mapsto [x_0 x_i^{d-1} : \dots : x_n x_i^{d-1}] = [x_0 : \dots : x_n],$$

the identity map. Equally easily, one checks that $\nu_d \circ \phi : \nu_d(\mathbb{P}^n) \to \mathbb{P}^n \to \nu_d(\mathbb{P}^n)$ is the identity map on W, that is, $z = \nu_d(\phi(z))$, so ν_d is surjective, and therefore $W \subset \nu_d(\mathbb{P}^n)$ and ν_d defines an isomorphism between $\mathbb{P}^n$ and $W = \nu_d(\mathbb{P}^n)$ with inverse ϕ.

□

Subvarieties of Veronese Varieties

Proposition 4.3. *If $Y \subseteq \mathbb{P}^n$ is a projective variety, then $\nu_d(Y)$ is a subvariety of $\nu_d(\mathbb{P}^n)$ and in particular of $\mathbb{P}^m$.*

Proof.
We defined $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ as an induced map from the map $\hat{v}_d : \mathbb{A}^{n+1} \to \mathbb{A}^{\binom{n+d}{d}}, [x_0 : \dots, : x_n] \mapsto [\dots : x^I : \dots]$ of the affine cones. Any $g(z) \in k[z_I : I \in \binom{n+d}{d}]$ of homogeneous degree a defines a map $g_a : \mathbb{A}^{\binom{n+d}{d}} \to k$. Then $g \circ \hat{\nu}_d$ has degree ad, and we can in fact get all homogeneous degree ad polynomials this way. For example, if $g = y_0$, then $a = 1$ and $g(\hat{\nu}_d(x)) = g(\dots : x^I : \dots) = x_0^d$.
Next, note that if $F \in k[x_0, \dots, x_n]$ is a homogeneous polynomial, then

$$\mathbb{V}(F) = \mathbb{V}(x_0 F, x_1 F, \dots, x_n F) \subseteq \mathbb{P}^n$$

holds. So if $Y = \mathbb{V}(F_1, \dots, F_r) \subseteq \mathbb{P}^n$, and m_i is the degree of F_i, and we choose a such that $ad > m_i$ for all i, then there exist homogeneous degree ad polynomials $G_1, \dots, G_s$ such that $Y = \mathbb{V}(G_1, \dots, G_s)$.
Then $G_i = H_i \circ \hat{v}_d$ for some H_i of degree a. Now by definition, $y \in \nu_d(Y)$ if and only if $y = \nu_d(x)$ where $G_i(x) = 0$ for all i. But $G_i(x) = H_i(\nu_d(x))$, so $x \in Y$ if and only if $\nu_d(x) \in \mathbb{V}(H_1, \dots, H_s)$. Therefore $\nu_d(Y) = \nu_d(\mathbb{P}^n) \cap \mathbb{V}(H_1, \dots, H_s)$. □

Example. $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ and $Y = \mathbb{V}(x_0^3 + x_1^3) \subseteq \mathbb{P}^2$. Then we can find homogeneous degree 4 generators using the trick in the proof:

$$Y = \mathbb{V}(x_0^4 + x_0 x_1^3, x_0^3 x_1 + x_1^4, x_0^3 x_2 + x_1^3 x_2)$$

where
$$x_0^4 + x_0x_1^3 = (x_0^2)^2 + (x_0x_1)x_1^2 = z_{(2,0,0)}^2 + z_{(1,0,0)}z_{(0,2,0)}$$
. Similarly we can rewrite the other two generator as homogeneous degree 2 polynomials on $\mathbb{P}^5$ to get

$$\nu_2(Y) = \mathbb{V}(z_{(2,0,0)}^2 + z_{(1,1,0)}z_{(0,2,0)}, z_{(0,2,0)}^2 + z_{(1,1,0)}z_{(2,0,0)}, z_{(2,0,0)}z_{(1,0,1)} + z_{(0,1,1)}z_{(0,2,0)})$$

Remark 4.4. Proposition 4.3 allows us to construct different embeddings of $Y \subset \mathbb{P}^n$ into big projective spaces with isomorphic images.

4.2 Segre maps

Recall from the first lecture, that the Zariski topology on $\mathbb{P}^2$ is not identical with the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$. To resolve this problem we describe $\mathbb{P}^1 \times \mathbb{P}^1$ as a subvariety of a bigger projective space and restrict the Zariski topology on the projective space to the subvariety.

Definition 4.3. The Segre morphism is defined as

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

$$([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) \mapsto [x_0y_0 : x_1y_0 : \ldots : x_ny_m]$$

where the coordinates run over the pairwise products. The Segre variety is $\Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$.

Remark. 1. We can view this map as matrix multiplication

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}(M_{(n+1)\times(m+1)})$$

$$((x_0, \ldots, x_n), (y_0, \ldots, y_m)) \mapsto \left[\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_0 & \ldots & y_m \end{pmatrix} \right] = [x_iy_j]$$

The coordinates z_{ij} on the target space $\mathbb{P}^{(n+1)(m+1)-1}$ are indexed by the entries of this $n \times m$ matrix.

2. An other interpretation is

$$\sigma_{n,m} : \mathbb{P}(\mathbb{A}^{n+1}) \times \mathbb{P}(\mathbb{A}^{m+1}) \to \mathbb{P}(\mathbb{A}^{n+1} \otimes \mathbb{P}^{m+1})$$

$$([x], [y]) \mapsto [x \otimes y]$$

3. The degree d Veronese embedding is the composition of the diagonal embedding of $\mathbb{P}^n$ into the product of d copies of $\mathbb{P}^n$, the Segre embedding and the projection from the tensor product to the symmetric product:

$$\mathbb{P}(\mathbb{A}^{n+1}) \hookrightarrow \underbrace{\mathbb{P}(\mathbb{A}^{n+1}) \times \ldots \times \mathbb{P}(\mathbb{A}^{n+1})}_{d} \to \mathbb{P}((\mathbb{A}^{n+1})^{\otimes d}) \to \mathbb{P}(\mathrm{Sym}^d(\mathbb{A}^{n+1}))$$

Proposition 4.4. ***i***) *$\Sigma_{n,m}$ is a projective variety cut out by the 2×2 minors of the matrix (z_{ij}). That is*

$$\Sigma_{n,m} = \{[z_{ij}] : 2 \times 2 \textit{ minors of } (z_{ij}) \textit{ vanish}\} = \mathbb{V}(z_{ij}z_{kl} - z_{il}z_{kj} : 0 \leq i < k \leq n, 0 \leq j < l \leq m)$$

ii) *Define the morphism $\pi_n : \Sigma_{n,m} \to \mathbb{P}^n$, ($\pi_m : \Sigma_{n,m} \to \mathbb{P}^m$) such that π_n takes $[z_{00}, \ldots, z_{nm}]$ to a non zero column $[z_{0j} : \ldots : z_{nj}] \in \mathbb{P}^n$ of the matrix. (Likewise π_m for the rows.) These are well defined morphisms.*

iii) *$\pi_n \times \pi_m : \Sigma_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m$ is the inverse morphism to $\sigma_{n,m}$.*

Proof.
i) The columns of $\sigma_{n,m}([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [x_i y_j]$ are scalar multiples of each other, so all 2×2 minors vanish on $\Sigma_{n,m}$. On the other hand, if all 2×2 minors of (z_{ij}) vanish then it has rank 0, or 1. Rank 0 is impossible though, since at least one $z_{ij} \neq 0$. So all columns are scalar multiples of one another. Let $x_0, \ldots, x_n$ be a nonzero column, and $y_0, \ldots, y_m$ be these scalars (at least one of them is nonzero) then $[z_{ij}] = \sigma_{n,m}([x_0, \ldots, x_n], [y_0, \ldots, y_m])$ is in the image of $\sigma_{n,m}$.
ii) Let $U^\Sigma_{ij} = \{[z_{00} : \ldots : z_{nm}] \in \Sigma_{n,m} | z_{ij} \neq 0\} = U_{ij} \cap \Sigma_{n,m}$ be the open subset where $z_{ij} \neq 0$. Define

$$\pi^{ij}_n : U^\Sigma_{ij} \to \mathbb{P}^n$$
$$[z_{00} : \ldots : z_{nm}] \mapsto [z_{0j} : \ldots : z_{nj}]$$

Since columns are scalar multiples of one another, $\pi^{ij}_n|U^\Sigma_{ij} = \pi^{kl}_n|U^\Sigma_{kl}$ for any i, j, k, l, so these define a morphism $\pi_n : \Sigma_{n,m} \to \mathbb{P}^n$.

iii) On the affine chart $U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$ where $y_i \neq 0$ and $x_j \neq 0$ the composition $(\pi_n \times \pi_m) \circ \sigma_{n,m}$ reads as

$$\begin{aligned}([x_0 : \ldots : x_n], [y_0, \ldots : y_m]) &\mapsto \left[\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_0 & \ldots & y_m \end{pmatrix}\right] \\ &\mapsto ([y_i x_0 : \ldots : y_i x_n], [x_j y_0 : \ldots : x_j y_m]) = ([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]),\end{aligned}$$

so on every open chart the composition is the identity, so $(\pi_n \times \pi_m) \circ \sigma_{n,m} = id_{\mathbb{P}^n \times \mathbb{P}^m}$. Similarly, on U_{ij}^{Σ} we have

$$(\sigma_{n,m} \circ (\pi_n \times \pi_m))([z_{00} : \ldots : z_{nm}]) = [z_{0j}z_{i0} : z_{1j}z_{i0} : \ldots : z_{nj}z_{im}] = [z_{00} : \ldots : z_{nm}]$$

using that the row and column rank of (z_{ij}) is 1. So $\sigma_{n,m} \circ (\pi_n \times \pi_m) = id_{\Sigma_{n,m}}$ □

Definition 4.5. The Zariski topology on $\mathbb{P}^n \times \mathbb{P}^m$ is defined as the induced topology on $\Sigma_{n,m}$. So $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \Sigma_{n,m}$ is an isomorphism with inverse $\pi_n \times \pi_m$ defined in Proposition 4.4.

Proposition 4.6. *If $Y \subseteq \mathbb{P}^n, Y \subset \mathbb{P}^m$ are projective varieties, then $\sigma_{n,m}(X \times Y) \subset \mathbb{P}^{(n+1)(m+1)-1}$ is a subvariety.*

Proof. We only give the equations of the image but do not prove that they indeed cut out the image. Suppose $X = \mathbb{V}(F_1, \ldots, F_s) \subseteq \mathbb{P}^n$ and $Y = \mathbb{V}(G_1, \ldots, G_r) \subseteq \mathbb{P}^m$. Then

$$\begin{aligned}\sigma_{n,m}(X \times Y) = \Sigma_{n,m} \cap \mathbb{V}(\{F_k(z_{0j}, \ldots, z_{nj}), G_l(z_{i0}, \ldots, z_{im}) : 1 \le k \le s, 1 \le l \le r,\\ 0 \le j \le m, 0 \le i \le n),\end{aligned}$$

so by definition $X \times Y$ is closed. □

Definition 4.7. If $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are projective varieties, then the topology on $X \times Y$ is the induced topology on $\sigma_{n,m}(X \times Y)$.

4.3 Grassmannians and Flag Manifolds

Grassmannians and flag manifolds are generalisations of projective spaces and they are fundamental objects in geometry and representation theory.

Definition 4.8. For $1 \le d < n$ the *Grassmannian* $\mathrm{Grass}(d, n)$ is the set of all d-dimensional vector subspaces of k^n:

$$\mathrm{Grass}(d, n) = \{V \subset k^n : V \text{ is a linear vector subspace of dimension } d\}$$

For $1 \le d_1 < d_2 < \ldots < d_s < n$ the *flag manifold* $\mathrm{Flag}(d_1, \ldots, d_s, n)$ is the set of flags of linear vector subspaces of dimension $d_1, \ldots, d_s$:

$$\mathrm{Flag}(d_1, \ldots, d_s, n) = \{V_1 \subset \ldots \subset V_s \subset k^n : V_i \text{ is a linear vector subspace of dimension } d_i\}$$

Theorem 4.9. *The Grassmannian $\mathrm{Grass}(d, n)$ can be embedded as a subvariety of $\mathbb{P}_k^{\binom{n}{d}-1}$.*

PROOF. Let $V \in \mathrm{Grass}(d,n)$ be a d dimensional subspace in k^n. Choose basis vectors $(a_{i1}, \dots a_{in})$, $i = 1, \dots, d$ of V and put these into the rows of a $d \times n$ matrix:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ a_{d1} & \cdots & a_{dn} \end{pmatrix}.$$

This matrix has full rank as its rows are linearly independent, and two matrices (a_{ij}) and (b_{ij}) define the same subspace if and only if there is a $g \in \mathrm{GL}(d)$ such that $(a_{ij}) = g \cdot (b_{ij})$, therefore

$$\mathrm{Grass}(d,n) = \{d \times n \text{ matrices of rank } d\}/\mathrm{GL}(d,k)$$

can be identifies by the orbits of this action. Let $\Delta_{i_1,\dots,i_d}$ denote the minor formed by the columns $1 \leq i_1 < \ldots < i_d \leq n$. Define the map $\mathrm{Grass}(d,n) \to \mathbb{P}_k^{\binom{n}{d}-1}$ by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ a_{d1} & \cdots & a_{dn} \end{pmatrix} \mapsto [\Delta_{1,\dots,d} : \ldots : \Delta_{n-d+1,n-d+2,\dots,n}],$$

which is well defined on the orbits of $\mathrm{GL}(d,k)$ as the action of g multiplies each minor by $\det(g)$, and at least one minor is nonzero (because the matrix has full rank). This map is injective (left as an exercise). This embedding is also known as the Plucker embedding. We can't prove here that the image is closed in $\mathbb{P}_k^{\binom{n}{d}-1}$, but its vanishing ideal is generated by the following Plucker relations: for any $l < d$

$$\Delta_{i_1,\dots,i_d}\Delta_{j_1,\dots,j_d} - \Sigma_{q_1<\ldots<q_l}\Delta_{i_1,\dots,j_1,\dots,j_l,\dots,i_d}\Delta_{i_{q_1},\dots,i_{q_l},j_{l+1},\dots,j_d}$$

where $(i_1, \dots, j_1, \dots, j_l, \dots, i_d)$ denotes the list $(i_1, \dots, i_d)$ with the entries $i_{q_1}, \dots, i_{q_l}$ replaced by $j_1, \dots, j_l$ and vice versa for the other factor.
For the details we recommend Harris: Algebraic geometry, a first course. □

REMARK 4.10.

1. For $k = \mathbb{C}$ an other interpretation of the Plucker embedding is the following:

$$\mathrm{Grass}(d,n) \to \mathbb{P}(\wedge^d \mathbb{C}^n) = \mathbb{P}^{\binom{n}{d}-1}$$
$$V \mapsto \wedge^d V$$

where $\wedge^d \mathbb{C}^n$ is the dth exterior power of $\mathbb{C}^n$. If $\{v_1, \dots, v_d\}$ is a basis of V then this map send V to the line spanned by $v_1 \wedge \ldots \wedge v_n$. For details we refer again to Harris' book.

2. For $k = \mathbb{C}$ the image of the Plucker embedding is smooth and $\mathrm{Grass}(d,n)$ is a complex submanifold of $\mathbb{P}_{\mathbb{C}}^{\binom{n}{d}-1}$. We do not prove this here.

3. It is not hard to find out the analogous Plucker embedding of the flag manifold

$$\mathrm{Flag}(d_1, \ldots, d_s, n) \hookrightarrow \mathbb{P}^{\binom{n}{d_1}-1} \times \ldots \times \mathbb{P}^{\binom{n}{d_s}-1}.$$

5 Hilbert's Nullstellensatz

In this section we study functions on affine and projective varieties. We will see that there is a strong relationship between the category of varieites and the algebra of functions on them.

5.1 Affine Nullstellensatz

Recall that if $X \subseteq \mathbb{A}^n$ an affine variety, then

$$\mathbb{I}(X) = \{f \in k[x_1, \ldots, x_n] : f(X) = 0\}$$

is the vanishing ideal of the variety. Note also that $\mathbb{I}(\mathbb{V}(I)) \neq I$ in general, for example $\mathbb{I}(\mathbb{V}(x^2)) = (x)$.

Lemma 5.1. *Let X, Y be affine varieties in $\mathbb{A}^n$, I, J ideals in $k[x_1, \ldots, x_n]$*

1. $X \subset Y \Rightarrow \mathbb{I}(X) \supset \mathbb{I}(Y)$
2. $I \subset J \Rightarrow \mathbb{V}(I) \supset \mathbb{V}(J)$
3. *X an affine variety, then* $X = \mathbb{V}(\mathbb{I}(X))$
4. *I an ideal, then* $I \subset \mathbb{I}(\mathbb{V}(I))$

Proof.
1) $f(Y) = 0 \Rightarrow f(X) = 0$.
2) If $x \in \mathbb{V}(J)$ then $f(x) = 0$ for all $f \in J \Rightarrow f(x) = 0$ for all $f \in I$ which implies $x \in \mathbb{V}(I)$.
3) $x \in X$ then $f(x) = 0$ for all $f \in \mathbb{I}(X)$ so $X \subseteq \mathbb{V}(\mathbb{I}(X))$. On the other hand $X = \mathbb{V}(J)$ for some ideal J. Then $J \subseteq \mathbb{I}(X)$. In this case though, we get from (2) that $\mathbb{V}(\mathbb{I}(X)) \subseteq \mathbb{V}(J) = X$
4) If $f \in I$ then $f(x) = 0$ for all $x \in \mathbb{V}(I)$. □

Definition 5.2. For $J \lhd k[x_1, \ldots, x_n]$, the *radical* of J is defined as

$$\sqrt{J} = \left\{ r \in k[x_1, \ldots, x_n] : \exists k \geq 1, r^k \in J \right\}.$$

$\sqrt{J}$ is an ideal: if $r_1^{k_1}, r_2^{k_2} \in J$ then $(r_1+r_2)^{k_1+k_2} = \sum \binom{k_1+k_2}{i} r_1^i r_2^{k_1+k_2-i} \in J$ as $i \geq k_1$ or $k_1+k_2-i \geq k_2$ holds. And if $r^k \in J, f \in k[x_1,\ldots,x_n]$ then $(rf)^k = r^k f^k \in J$.
For example, $(x^2) \subset k[x]$ is not equal to its radical ideal, its radical is $\sqrt{(x^2)} = (x)$.

Definition 5.3. Let R be a commutative ring with unit.

- We call an ideal $J \subset R$ radical ideal, if $J = \sqrt{J}$.
- $\mathfrak{p} \subsetneq R$ is prime ideal if $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
- $\mathfrak{m} \subsetneq R$ is maximal ideal if the only ideal strictly containing it is R.

Remark. Recall from commutative algebra the following results.

1. Maximal $\Rightarrow$ prime $\Rightarrow$ radical for ideals in any ring. Note also that radical doesn't necessarily imply prime, for example $(xy) \subset k[x,y]$ is radical but not prime. And prime does not imply maximal, e.g. $(y-x^2) \subset k[x,y]$.
2. $I \subset R$ is maximal $\Leftrightarrow R/I$ is a field.
   When $R = k[x_1,\ldots,x_n]$ we can say more (see Theorem 5.9):
   $I \subset k[x_1,\ldots,x_n]$ is maximal $\Leftrightarrow k[x_1,\ldots,x_n]/I \simeq k$ is the base field.
3. $I \subset R$ is prime ideal $\Leftrightarrow R/I$ is an integral domain. (no zero divisors)

Theorem 5.4. *(Nullstellensatz) Let k be an algebraically closed field. Then*

1. *Maximal ideals of $k[x_1,\ldots,x_n]$ are of the form $(x-a_1,\ldots,x-a_n) = \mathbb{I}(a)$ where $a = (a_1,\ldots,a_n) \in k^n$.*
2. *If $J \subsetneq k[x_1,\ldots,x_n]$ is a proper ideal, then $\mathbb{V}(J) \neq \emptyset$*
3. *We have that $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$*

Note that 2) is not true for $k = \mathbb{R}$: $\mathbb{V}(x^2+1) = \emptyset$ in $\mathbb{A}^1_{\mathbb{R}}$.

Lemma 5.5. $\mathbb{I}(X)$ *is a prime ideal if and only if X is irreducible*

Proof.
If X is irreducibe and $ab \in \mathbb{I}(X)$ but $a \notin \mathbb{I}(X), b \notin \mathbb{I}(X)$ then $X \subset \mathbb{V}(a) \cup \mathbb{V}(b)$ so $X = (X \cap \mathbb{V}(a)) \cup (X \cap \mathbb{V}(b))$ is a nontrivial decomposition as $a \notin \mathbb{I}(X)$ and therefore $X \cap \mathbb{V}(a) \subsetneq X$ and similarly $X \cap \mathbb{V}(b) \subsetneq X$. A contradiciton.
Assume $\mathbb{I}(X)$ is prime but $X = (X \cap \mathbb{V}(I_1)) \cup (X \cap \mathbb{V}(I_2))$ is a nontrivial decomposition. Since $X_1 \nsubseteq X_2$, $I_1 \nsubseteq I_2$, and therefore $\exists p \in I_2 \setminus I_1$, and similarly $q \in I_1 \setminus I_2$. Then $pq \in I_1 I_2 = \mathbb{I}(X_1 \cup X_2) = \mathbb{I}(X)$ but $p, q \notin I_1 I_2 = \mathbb{I}(X)$ so $\mathbb{I}(X)$ is not prime, contradiction. $\square$

Corollary 5.6. *According to the Nullstellensatz, we have the following bijections*

$$\begin{array}{rcl}
\{\textit{Radical ideals in } k[x_1,\dots,x_n]\} & \leftrightarrow & \{\textit{Affine subvarieties of } \mathbb{A}^n_k\} \\
\cup & & \cup \\
\{\textit{Prime ideals}\} & \leftrightarrow & \{\textit{Irreducible subvarieties of } \mathbb{A}^n_k\} \\
\cup & & \cup \\
\{\textit{Maximal ideals}\} & \leftrightarrow & \{\textit{Points in } \mathbb{A}^k_n\}
\end{array}$$

We will need 3 ingredients to the proof of the Nullstellensatz.

Lemma 5.7. *A ring R is Noetherian (that is, all ideals are finitely generated) if any ascending sequence of ideals terminates. That is, if $I_0 \subseteq I_1 \subseteq \cdots \subset R$, then there exists an n such that $I_n = I_{n+1} = \dots$.*

Definition 5.8. A topological space is Noetherian if all descending sequences of closed sets terminate. That is, if $X_0 \supset X_1 \supset \cdots$ is a chain of closed subsets then $X_n = X_{n+1} = \dots$ for some n.

We don't prove the following technical theorem here, for a proof see Miles Reid's Undergraduate Algebraic Geometry.

Theorem 5.9. *If k is a field with infinitely many elements and if K is another field, finitely generated over k as a k-algebra, then K is algebraic over k.*

We turn to the proof of the Nullstellensatz.
Proof. To prove (1) first we show that if $a = (a_1,\dots,a_n) \in \mathbb{A}^n_k$, then $\mathbb{I}(a) = (x_1 - a_1,\dots,x_n - a_n) \subset k[x_1,\dots,x_n]$ is a maximal ideal. Define the evaluation map at a:

$$\begin{array}{rcl}
\pi : k[x_1,\dots,x_n] & \to & k \\
g & \mapsto & g(a)
\end{array}$$

This is clearly onto, so

$$\frac{k[x_1,\dots,x_n]}{Ker\pi} \cong k$$

and so $Ker\pi$ is maximal. But $x_i - a_i \in Ker\pi$ for all i, so $(x_1 - a_1,\dots,x_n - a_n) \subseteq Ker\pi$ On the other hand, applying the affine linear transformation $x_i \mapsto x_i - a_i$, there is an expansion around $a = (a_1,\dots,a_n)$:

$$g(x_1,\dots,x_n) = g(a_1,\dots,a_n) + \sum \alpha(x_i - a_i) + \sum \alpha\beta(x_i - a_i)(x_j - a_j) + \dots$$

Thus, if $g(a_1,\dots,a_n) = 0$ then $g \in (x_1 - a_1,\dots,x_n - a_n)$, so $Ker\pi \subset (x_1 - a_1,\dots x_n - a_n)$ is maximal, and therefore they are equal and maximal.

Conversely, suppose $\mathfrak{m}$ is a maximal ideal in $k[x_1, \dots, x_n]$, and we show that $\mathfrak{m} = \mathbb{I}(a)$ for some $a \in \mathbb{A}^n$. First,

$$k[x_1, \dots, x_n]/\mathfrak{m} := K$$

is a field. K is finitely generated as a k-algebra, so apply Theorem 5.9. This implies that K is algebraic over k, and therefore $k = K$ (since k is algebraically closed). Let π denote the projection to the the quotient. Then we have

$$k \hookrightarrow k[x_1, \dots, x_n] \xrightarrow[\pi]{Ker\pi=\mathfrak{m}} K = k$$

Define $a_i = \pi(x_i) \in k$. Since π is a projection, we have $\pi^2 = \pi$, so $\pi(\pi(x_i)) = \pi(x_i)$, and therefore $\pi(x_i - a_i) = \pi(x_i) - \pi(\pi(x_i)) = 0$, so $x_i - a_i \in Ker\pi$ for all $i = 1, \dots, n$. Therefore

$$(x_1 - a_1, \dots, x_n - a_n) \subseteq Ker\pi = \mathfrak{m},$$

but as we proved above, $(x_1 - a_1, \dots, x_n - a_n)$ is maximal, and if two maximal ideals contain each other they must be equal.

(1) $\Rightarrow$ (2) We have that $k[x_1, \dots, x_n]$ is Noetherian so chains of ascending ideals terminate. So for $J \subseteq k[x_1, \dots, x_n]$ we must have that $J \subseteq \mathfrak{m}$, where $\mathfrak{m}$ is some maximal ideal. This now gives that $\mathbb{V}(\mathfrak{m}) \subseteq \mathbb{V}(J)$ by Lemma 5.1. (1), however, implies that $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a \in \mathbb{A}^n_k$. Thus, $a \in \mathbb{V}(\mathfrak{m}) \subseteq \mathbb{V}(J)$ which implies that $\mathbb{V}(J) \neq \emptyset$

(2) $\Rightarrow$ (3) Suppose that $J \subseteq k[x_1, \dots, x_n]$. We want to show that $\sqrt{J} = \mathbb{I}(\mathbb{V}(J))$. Let $f \in \mathbb{I}(\mathbb{V}(J))$, want to show that $f^N \in J$ for some N. Introduce t, an extra variable, and define $J_f = (J, ft - 1) \subseteq k[x_1, \dots, x_n, t]$. Then,

$$\mathbb{V}(J_f) = \big\{(a_1, \dots, a_n, b) = (p, b) \in \mathbb{A}^{n+1} : p \in \mathbb{V}(J), bf(p) = 1\big\}$$

But if $bf(p) = 1$ then $f(p) \neq 0$. However, $p \in \mathbb{V}(J)$ and $f \in \mathbb{I}(\mathbb{V}(J))$ so $f(p) = 0$ by definition, which means $\mathbb{V}(J_f) = \emptyset$. Thus, (2) implies that $J_f = k[x_1, \dots, x_n, t]$ and $1 \in J_f$. Now, let $f_1, \dots, f_n$ be generators for J and $f_i \in k[x_1, \dots, x_n]$. We have that

$$1 = \sum_{i=1}^{n} a_i f_i + a_0(ft - 1)$$

for some $a_i \in k[x_1, \dots, x_n, t]$. Let N be the highest power of t appearing in the a_i's. Then $f^N \cdot a_i(x_1, \dots, x_n, t) = b_i(x_1, \dots, x_n, ft)$ where b_i are some other polynomials in $n + 1$ variables. So $f^N = \sum b_i f_i + b_0(ft - 1) \in k[x_1, \dots, x_n, ft]$. The image of this equation in $k[x_1, \dots, x_n, ft]/(ft - 1)$. is $\overline{f}^N = \sum \overline{b}_i \overline{f}_i \in k[x_1, \dots, x_n, ft]/(ft - 1)$. Note that

$$\iota : k[x_1, \dots . x_n] \hookrightarrow k[x_1, \dots, x_n, ft]/(ft - 1)$$

is an embedding, and

$$\overline{b}_i = \iota(b_i(x_1, \ldots, x_n, 1)), \overline{f}_i = \iota(f_i(x_1, \ldots, x_n))$$

are in the image. Since ι is an injection and homomorphism, the equation $\overline{f}^N = \sum \overline{b}_i \overline{f}_i$ implies $f^N = \sum b_i(x_1, \ldots, x_n, 1) f_i(x_1, \ldots, x_n) \in J$. $\square$

A fundamental idea in algebraic geometry is that affine varieties are in bijection with finitely generated reduced k-algebras.

Definition 5.10. If $X \subseteq \mathbb{A}^n$ is an affine variety, then

$$A(X) := k[x_1, \ldots, x_n]|_X \cong k[x_1, \ldots, x_n]/\mathbb{I}(X)$$

is the *coordinate ring* of X.

These are polynomial functions on X, think of $A(X)$ as $k[\overline{x}_1, \ldots, \overline{x}_n]$ where $\overline{x}_i = x_i + \mathbb{I}(X)$.

Considering $A(X)$ as a quotient ring, remember that the ideals J in R/I are in one to one correspondence with ideals $\widetilde{I}$ in R such that $J \subset \widetilde{I}$.

Definition. Let R be a commutative ring with unit. We call $0 \neq r \in R$ *nilpotent* if $r^n = 0$ for some n. A ring without nilpotent elements is called *reduced.*

Lemma 5.11. *$k[x_1, \ldots, x_n]/I$ is reduced if and only if I is radical.*

Proof. Note that $\sqrt{I} = I$ if and only if $f^n \in I$ implies that $f \in I$, but $f + I = 0$ in $k[x_1, \ldots, x_n]/I$ if and only if $f \in I$. $\square$

Therefore by Lemma 5.1, $A(X)$ is reduced for any affine variety X.

Example. Let $X = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$, then $A(X) = k[x, y]/(y - x^2) \cong k[x]$. Remember that $X \cong \mathbb{A}^1$ and $A(\mathbb{A}^1) = k[y]$, so isomorphic varieties in this example have isomorphic coordinate rings. Is this always true?

The map $X \to A(X)$ associates to an affine variety a finitely generated reduced algebra. Can we construct a a reverse map? Let R be a finitely generated reduced k-algebra. Let $a_1, \ldots, a_n$ be its generators. The map $k[x_1, \ldots, x_n] \to R$ such that $x_i \mapsto a$ is surjective. Let I be the kernel. We then have that $k[x_1, \ldots, x_n]/I \cong R$. R is reduced so I must be a radical ideal of $k[x_1, \ldots, x_n]$. We can take $\mathbb{V}(I) = X_R \subseteq \mathbb{A}^n$. This is not a well-defined correspondence yet, as we could have picked different generators, say $b_1, \ldots, b_m$. Would this have given the same variety though?

Theorem 5.12. *We have a contravariant equivalence of categories*

$$\left\{\begin{array}{c}\textit{affine varieties and}\\ \textit{morphisms of affine varieties}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\textit{finitely generated reduced}\\ k\textit{-algebras and homomorphisms of } k\textit{-algebras}\end{array}\right\}$$

This means that

1. *The two processes $X \to A(X)$ and $R \to X_R$ defined above are well defined, and inverse to each other, that is:*
   - *If X, Y are isomorphic varieties then $A(X) \simeq A(Y)$. If R, S are isomorphic algebras then X_R and X_S are isomorphic varieties.*
   - *$X \simeq X_{A(X)}$ and $R \simeq A(X_R)$.*
2. *If $F : X \to Y$ is a morphism of affine varieties, it induces a k-algebra homomorphism $F^\# : A(Y) \to A(X)$ and $(F \circ G)^\# = G^\# \circ F^\#$. (So $\#$ is a contravariant functor)*
3. *If $f : R \to S$ is a k-algebra homomorphism, then we have a morphism of affine varieties $f_\# : X_S \to X_R$ such that $(f \circ g)_\# = g_\# \circ f_\#$.*
4. *$\cdot_\#$ is the inverse functor of $\cdot^\#$, that is, $(F^\#)_\# = F$ and $(f_\#)^\# = f$ up to isomorphism, in other words the following two diagrams commute*

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\cong"] & Y \arrow[d, "\cong"] \\ X_{A(X)} \arrow[r, "(F^\#)_\#"] & Y_{A(Y)} \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\cong"] & S \arrow[d, "\cong"] \\ A(X_R) \arrow[r, "(f_\#)^\#"] & A(X_S) \end{tikzcd}$$

Let's see some examples before we start the proof.

Example. Let $X = \mathbb{V}(x^2 - y) \subseteq \mathbb{A}^2$ and $Y = \mathbb{A}^1$ and $I : X \to Y$ such that $(x, y) \mapsto y$. Then

$$\begin{array}{rcl} F^\# : k[y] & \to & k[x, y]/(x^2 - y) \cong k[x] \\ y & \mapsto & (y \circ F)(x, y) = y \end{array}$$

and this map is an isomorphism, so the parabola $\mathbb{V}(x^2 - y)$ is isomorphic to the x axis $\mathbb{A}^1$, the isomorphism is given by the projection. This is what we expected.

Example. Let $X = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$ and $Y = \mathbb{A}^1$ and $F : \mathbb{A}^1 \to X$ such that $t \mapsto (t^2, t^3)$. The pullback is defined as

$$\begin{array}{rcl} F^\# : k[x, y]/(y^2 - x^3) & \to & k[t] \\ x & \mapsto & x \circ F = t^2 \\ y & \mapsto & y \circ F = t^3 \end{array}$$

Here, t is not in the image and so $X \not\cong \mathbb{A}^1$. Indeed, $X = \mathbb{V}(y^2 - x^3)$ is the cusp, with singularity at the origin, which is heuristically not isomorphic to $\mathbb{A}^1$.

We turn to the proof now.
Proof. Proof of 2) Given a morphism $F : X \to Y$ and $g \in A(Y)$, g represents a polynomial function $g : Y \to k$. Then $g \circ F : X \to k$ is a polynomial function so define $F^\# : A(Y) \to A(X)$ such that $g \mapsto F^\# g = g \circ F$. It is then clear that $(G \circ F) = F^\# \circ G^\#$.
Proof of 3) If $f : R \to S$ is a k-algebra homomorphism then choose representations

$$R = \frac{k[x_1, \ldots, x_n]}{I} \text{ and } S = \frac{k[y_1, \ldots, y_m]}{J}$$

so we have $\mathbb{V}(I) \subseteq \mathbb{A}^n$ and $\mathbb{V}(J) \subseteq \mathbb{A}^m$. We want a morphism $\mathbb{V}(I) \to \mathbb{V}(J)$. We will define a morphism $f_\# : \mathbb{A}^m \to \mathbb{A}^n$ which satisfies $f_\#|_{V(J)} : \mathbb{V}(J) \to \mathbb{V}(I)$, giving us the desired morphism $\mathbb{V}(J) \to \mathbb{V}(I)$.
Let $F_i \in k[y_1, \ldots, y_m]$ be the polynomial representing $f(x_i) \in k[y_1, \ldots, y_m]/J$. Define $f_\# : \mathbb{A}^m \to \mathbb{A}^n$ by $f_\# : a \mapsto (F_1(a), \ldots, F_n(a))$. This map on $\mathbb{A}^m$ depends on our choice of the F_i's. Let $a \in \mathbb{V}(J)$, we want to show that $f_\#(a) \in \mathbb{V}(I)$. Let $G \in I$. Consider $G(f_\#(a)) = G(F_1(a), \ldots, F_n(a))$. Now $a \in \mathbb{V}(J)$ so any $p \in k[x_1, \ldots, x_m]/J$ is well defined on a and in particular, $f(x_i)$ is well defined and equal to the chosen representative F_i, that is, if F_i, F_i' represent the same element $f(x_i)$ in S then $F_i(a) = F_i'(a)$, and we denote this by $f(x_i)(a)$. So

$$G(f_\#(a)) \;=\; G(f(x_1)(a), \ldots, f(x_n)(a)) = f(G)(a)$$

(Here we use that f is an algebra homomorphism! Namely, $f(G(x_1, \ldots, x_n)) = G(f(x_1), \ldots, f(x_n))$.)
We have however that $G \in I$ and so $G = 0$ in R, so $f(G) = 0$ and $G(f_\#(a)) = 0$ for all $G \in I$, therefore $f_\#(a) \in \mathbb{V}(I)$. Now for

$$R = \frac{k[x_1, \ldots, x_n]}{I} \to^f S = \frac{k[y_1, \ldots, y_m]}{J} \to^g T = \frac{k[z_1, \ldots, z_l]}{L}$$

we have

$$(f \circ g)_\#(a) = ((f \circ g)(x_1)(a), \ldots (f \circ g)(x_n)(a)) = (g(f(x_1))(a), \ldots, g(f(x_n))(a)) =^*$$
$$(f(x_1)(g(y_1)(a), \ldots, g(y_m)(a)), \ldots, f(x_n)(g(y_1)(a), \ldots, g(y_m)(a))) = f_\#(g_\#(a)) = (g_\# \circ f_\#)(a),$$

where at $*$ we use again that f,g are algebra homomorphisms.
Proof of 1) Using the defined functors this is automatic now. If $F : X \to Y$ is an isomorphism, there is an inverse $F^{-1} : Y \to X$ such that $F \circ F^{-1} = Id$. Now $(Id_X)^\# = Id_{A(X)}$ by definition, so $id_{A(Y)} = (F \circ F^{-1})^\# = (F^{-1})^\# \circ F^\#$ so $F^\#$ has an inverse, and therefore it is an isomorphism.

Similarly, if $f : R \to S$ is an isomorphism of algebras with inverse f^{-1} then $Id_{X_S} = (f \circ f^{-1})_\# = f^{-1}_\# \circ f_\#$, so $f^{-1}_\#$ is an inverse of $f_\#$. Finally, $X \simeq X_{A(X)}$ and $R \simeq A(X_R)$ follow from the definitions.

Proof of 4) Choose representations $R = k[x_1, \ldots, x_n]/I, S = k[y_1, \ldots, y_m]/J$ and $a \in \mathbb{A}^m, g \in A(X_R) = k[x_1, \ldots, x_n]/I$. Then we have

$$\begin{aligned}(f_\#)^\#(g)(a) &= (g \circ f_\#)(a) \\ &= g(f(x_1)(a), \ldots, f(x_n)(a)) \\ &= f(g)(a)\end{aligned}$$

and so $(f_\#)^\#(g)(a) = f(g)(a)$ for all $a \in \mathbb{A}^m$. This means that $(f_\#)^\#(g)$ and $f(g)$ are the in S. □

5.2 Projective Nullstellensatz

Remember that projective varieties are defined by homogeneous polynomials. Now we study the homogeneous version of the affine Nullstellensatz.

Definition 5.13. A graded ring is a commutative ring of the form $R = \oplus_{d \geq 0} R_d$, where R_d is a subgroup under addition, $R_d \cap R_e = \{0\}$ when $d \neq e$, and $R_d R_e \subseteq R_{d+e}$ where $R_d R_e = \{r_d r_e : r_d \in R_d, r_e \in R_e\}$. We call the elements of R_i the degree i elements.

Example. $k[x_1, \ldots, x_n]$ with $k[x_1, \ldots, x_n]_d =$ degree-d homogeneous polynomials

Definition 5.14. $I \subseteq R$ is a homogeneous ideal of the graded ring R if $I = \oplus_{d \geq 0} I \cap R_d$ holds.

Proposition 5.15.

1. *I is a homogeneous ideal if and only if I can be generated by homogeneous elements.*
2. *If I is homogeneous then I is prime if and only if for all f, g homogeneous in R, $fg \in I$ implies that $f \in I$ or $g \in I$*
3. *Sums, products, intersections, radicals of homogeneous ideals are homogeneous.*
4. *If R is graded and I is homogeneous, then R/I is a graded ring.*

Proof. Exercise 5. on Sheet 3. □

Recall that for affine varieties $X \subseteq \mathbb{A}^{n+1}$, we have $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$ and an isomorphism of categories

$$\{\text{affine varieties, morphisms of affine varieties}\} \simeq$$
$$\{\text{finitely generated reduced } k\text{-algebras, k-algebra homomorphisms}\}$$

For projective varieties, the story is not quite as good but there is still a lot to be said. Projective varieties are defined by homogeneous ideals, that is $X = \mathbb{V}(I)$ where I is generated by homogeneous elements.

Definition 5.16. If $X \subseteq \mathbb{P}^n$ is a projective variety then $\mathbb{I}(X) = \{F \in k[x_0, \ldots, x_n] | F$ is homogeneous $, F($
$0\}$ is the homogeneous vanishing ideal of X.

We have $\mathbb{V}(x_0, \ldots, x_n) = \emptyset$, so the Nullstellensatz is not true for homogeneous ideals. We call $(x_0, \ldots, x_n) \subset k[x_0, \ldots, x_n]$ the **irrelevant ideal**. In the affine case, it corresponds to $\{0\} \in \mathbb{A}^{n+1}$, which is not a point of $\mathbb{P}^n$. Lemma 5.1 remains true in projective case, and the proof is the same:

Lemma 5.17.
1. *$I \subseteq J \Rightarrow \mathbb{V}(J) \subseteq \mathbb{V}(I)$ for I, J homogeneous ideals*
2. *$X \subseteq Y \Rightarrow \mathbb{I}(Y) \subseteq \mathbb{I}(X)$ for X, Y projective varieties in $\mathbb{P}^n$*
3. *$X = \mathbb{V}(\mathbb{I}(X))$ for X a projective variety in $\mathbb{P}^n$*
4. *$I \subseteq \mathbb{I}(\mathbb{V}(I))$ for I a homogeneous ideal in $k[x_1, \ldots, x_n]$*

But due to the unwanted irrelevant ideal, the Nullstellensatz is slightly different:

Theorem 5.18. *(Homogeneous Nullstellensatz) Let k be algebraically closed field. Then*

1. *If J is a homogeneous ideal then $\mathbb{V}(J) = \emptyset \subset \mathbb{P}^n$ if and only if $(x_0, \ldots, x_n) \subset \sqrt{J}$*
2. *If $\mathbb{V}(J) \neq \emptyset$ then $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$*

Proof. This is a corollary of the affine case. Write $X = \mathbb{V}(J) \subset \mathbb{P}^n$, and $\hat{X} = \mathbb{V}(J) \subset \mathbb{A}^{n+1}$ for the affine cone over $\mathbb{V}(J)$. Then

1. $X = \emptyset \Leftrightarrow \hat{X} \subseteq \{0\} \Leftrightarrow \mathbb{I}(\hat{X}) = \sqrt{J} \supseteq \mathbb{I}(\{0\}) = (x_0, \ldots, x_n)$ by the affine Nullstellensatz.
2. Suppose that $X = \mathbb{V}(J) \neq \emptyset$. Then $\hat{X} \subset \mathbb{A}^{n+1}$ is defined and we have that

$$F \in \mathbb{I}(X) \Leftrightarrow F(X) = 0 \Leftrightarrow F(\hat{X}) = 0 \Leftrightarrow F \in \mathbb{I}(\hat{X}) = \sqrt{J}$$

   by the affine Nullstellensatz. Where did we use that $X \neq \emptyset$?

$\square$

Corollary 5.19. *All the arrows below are* $1:1$

$$\begin{array}{rcl}
\{\textit{Homogeneous radical ideals } J \subsetneq k[x_0,\dots,x_n]\} & \leftrightarrow & \{\textit{Projective varieties } V \subseteq \mathbb{P}^n\} \\
\cup & & \cup \\
\{\textit{Homogeneous prime ideals}\} & \leftrightarrow & \{\textit{Irreducible projective varieties } V \subseteq \mathbb{P}^n\} \\
\cup & & \cup \\
\{\textit{Irrelevant ideal}(x_0,\dots,x_n)\} & \leftrightarrow & \{\textit{Empty set}\}
\end{array}$$

Now we prove our theorem on the projective closure of an affine variety.

Theorem 5.20. *Identify* $\mathbb{A}^n$ *with* $U_0 \subseteq \mathbb{P}^n$ *(*$U_0 = \{[x_0 : \ldots : x_n] : x_0 \neq 0\}$*). Let* $X \subseteq \mathbb{A}^n$ *be an affine variety. Let* $\widetilde{I}$ *be the ideal generated by the homogenisation of all elements in* $\mathbb{I}(X)$*. Then* $\widetilde{I}$ *is radical and* $\mathbb{V}(\widetilde{I}) = \overline{X} \subseteq \mathbb{P}^n$*.*

Proof. First we'll show $\overline{X} \subseteq \mathbb{V}(\widetilde{I})$. If $G \in \widetilde{I}$ then $G|_{U_0}(x_0,\dots,x_n) = G(1,x_1,\dots,x_n) \in \mathbb{I}(X)$. So $G|_{U_0}(X) = 0$. Hence, $G|_{U_0\cap X} = G|_X = 0$. But G vanishes on some closed set $\mathbb{V}(G) \subset \mathbb{P}^n$ and $\overline{X}$ is the smallest closed set in $\mathbb{P}^n$ containing X. So $G(\overline{X}) = 0$. We now show that $\mathbb{V}(\widetilde{I}) \subseteq \overline{X}$. This is implied by $\mathbb{I}(\overline{X}) \subseteq \mathbb{I}(\mathbb{V}(\widetilde{I}))$. But $\widetilde{I} \subseteq \mathbb{I}(\mathbb{V}(\widetilde{I}))$ so it's enough to show that $\mathbb{I}(\overline{X}) \subseteq \widetilde{I}$ ($\Rightarrow \mathbb{V}(\widetilde{I}) \subseteq \mathbb{V}(\mathbb{I}(\overline{X})) = \overline{X}$). Suppose $G \in k[x_0,\dots,x_n]$ homogeneous and $G(\overline{X}) = 0$ ($G \in \mathbb{I}(\overline{X})$), then $G(X \cap U_0) = 0$ so $G(1,x_1,\dots,x_n) := g(x_1,\dots,x_n) \in \mathbb{I}(X)$. So the homogenisation $\widetilde{g}$ of g is in $\widetilde{I}$. Then $\widetilde{g}x_0^t = G$ fr some $t \geq 0$. So $G \in \widetilde{I}$. Hence, $\mathbb{I}(\overline{X}) \subseteq \widetilde{I}$. Finally we have seen that $\mathbb{I}(\mathbb{V}(\widetilde{I})) \subseteq \widetilde{I}$. The lemma tells us that $\widetilde{I} \subseteq \mathbb{I}(\mathbb{V}(\widetilde{I}))$ so $\widetilde{I} = \mathbb{I}(\mathbb{V}(\widetilde{I})) = \sqrt{\widetilde{I}}$ by the Nullstellensatz. □

5.3 Do we get equivalence of categories?

Recall that elements of the affine coordinate ring are polynomial functions on X. We don't have polynomial functions on projective varieties (just the constant functions). We do however have $\mathbb{I}(X) \subseteq k[x_0,\dots,x_n]$.

Definition 5.21. The homogeneous coordinate ring of $X \subset \mathbb{P}^n$ is the coordinate ring of its cone:

$$S(X) = k[x_0,\dots,x_n]/\mathbb{I}(X) = A(\hat{X})$$

Note that there is no interpretation of polynomials on X. These are polynomials on $\hat{X}$.
So we have a map $X \mapsto S(X)$, and now we want a map in the other direction like in the affine case: given a graded finitely generated k-algebra, can we get a projective variety? We want a representation $R = k[x_0,\dots,x_n]/I$ for some homogeneous ideal I. But the degree of $x_1,\dots,x_n$ is 1, so to get an isomorphism of graded algebras R need to have a set of generators of degree 1. Then write $R = k[x_0,\dots,x_n]/I$ and let $X_R = \mathbb{V}(I)$. Hence, we have the following equivalence

Theorem 5.22. *The homogeneous nullstellensatz defines a bijection*

$$\{\textit{Projective Varieties with an embedding } X \subset \mathbb{P}^n\} \leftrightarrow \left\{\begin{array}{c}\textit{Reduced, finitely generated } k\textit{-algebras} \\ \textit{generated by } n{+}1 \textit{ elements of degree 1} \\ \textit{with a representation}\end{array}\right\}$$

Remark. How much weaker is this than the affine case?

- $X \subseteq \mathbb{P}^n$ comes with an embedding.
- k-algebra must have the right number of generators of degree 1.
- No morphism picture here, see the coming remark below. Therefore, this is not an equivalence of categories.

Remark. The above correspondence we have is really the correspondence between affine cones of projective varieties and reduced finitely generated in degree 1 k-algebras. But a morphism $\widetilde{X} \to \widetilde{Y}$ of affine cones, does not necessarily descend everywhere to a morphism $X \to Y$ of projective varieties. (See Ex. 3 on Sheet 1).
Moreover, isomorphic projective varieties can have non-isomorphic affine cones, so $S(X)$ is not invariant under isomorphism. Here is an example.
If R is a graded ring, say $R = \oplus_{d\geq 0} R_d$, we may define $R^{(e)} = \oplus_{d\geq 0} R_{de}$, give grading $R^{(e)}_d = R_{de}$. Then (if R is a reduced, finitely generated in degree 1 k-algebra), then R and $R^{(e)}$ define the same projective variety. (Proof on problem sheet 4). But R is not isomorphic to $R^{(e)}$ in general as graded algebra, since they have different number of generators in degree 1.

Example. Take the Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}}$. We have proved that $\nu_d(\mathbb{P}^n) \cong \mathbb{P}^n$ as projective varieties. But their affine cone is not isomorphic! The affine cones are determined by the graded rings $k[x_0, \ldots, x_n]$ and $k[x_0, \ldots, x_n]^{(d)}$, and the first has $n+1$, the latter has $\binom{n+d}{d}$ generators of degree 1.

5.4 Spectrum, Maximal Spectrum and Schemes

Recall that points in affine space are in bijection with maximal ideals in $k[x_1, \ldots, x_n]$, namely $\{a\} = \mathbb{V}(x_1 - a_1, \ldots, x_n - a_n)$ for $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$.

Definition 5.23. The maximal spectrum

$$Spec_m k[x_1, \ldots, x_n] = \{m \subseteq k[x_1, \ldots, x_n] : m \text{ is a maximal ideal}\}$$

As a set $Spec_m k[x_1, \ldots, x_n] = \mathbb{A}^n$. Can we put a topology on $Spec_m k[x_1, \ldots, x_n]$ to have this as equality of topological spaces? Yes, we can.

Closed subsets of $\mathbb{A}^n$ are $\mathbb{V}(I)$'s such that I is an ideal of $k[x_1, \dots, x_n]$ and for $a \in \mathbb{V}(I)$ we have $I \subseteq (x_1 - a_1, \dots, x_n - a_n) = \mathbb{I}(\{a\})$. Thus, let

$$\mathbb{V}(I) = \{m \in Spec_m k[x_1, \dots, x_n] : I \subseteq m\}$$

We then have a topology on $Spec_m k[x_1, \dots, x_n]$ whose closed sets are $\{\mathbb{V}(I) : I \triangleleft k[x_1, \dots, x_n]\}$. We get

Theorem. $\mathbb{A}^n = Spec_m k[x_1, \dots, x_n]$ *as topological spaces*

Now let R be a finitely generated reduced k-algebra. Write $R = k[x_1, \dots, x_n]/\mathbb{I}(X)$ so $R = A(X)$. We have bijections

$$\begin{aligned} \text{Points in } X = \mathbb{V}(\mathbb{I}(X)) &\leftrightarrow \text{ max ideals } m \subseteq k[x_1, \dots, x_n] \text{ such that } \mathbb{I}(X) \subseteq m \\ &\leftrightarrow \text{ max ideals } \overline{m} \subseteq k[x_1, \dots, x_n]/\mathbb{I}(X) = R \end{aligned}$$

Define

$$Spec_m R = \{m \triangleleft R : m \text{ maximal}\}$$

Then $Spec_m R = X = X_R$ as a set with the identification $X \ni a \mapsto \mathfrak{m} = \{f \in R : f(a) = 0\} \subset k[x_1, \dots, x_n]$, and the inverse is $m \mapsto \mathbb{V}(m)$ for $m \subset R$ maximal.
Again, we want this bijection to be homeomorphism of topological spaces, so we put topology on $Spec_m R$. If $Y \subseteq X$ we have that $\mathbb{I}(X) \subseteq \mathbb{I}(Y)$ and so $\overline{\mathbb{I}(X)}$ is an ideal in $R = k[x_1, \dots, x_n]/\mathbb{I}(X)$, and points of $Y \subseteq \mathbb{A}^n$ are in bijection with maximal ideal $m \in k[x_1, \dots, x_n]$ such that $\mathbb{I}(Y) \subseteq m$ which is in bijection with maximal ideals $\overline{m} \subseteq k[x_1, \dots, x_n]/\mathbb{I}(X)$ such that $\overline{\mathbb{I}(Y)} \subseteq m$.
So, for $I \triangleleft R$, define $\mathbb{V}(I) = \{m \in Spec_m R | I \subseteq m\}$, and define these to be the closed subsets of $Spec_m(R)$.

Theorem. $Spec_m(R) = X_R$ *as toplogiacal spaces.*

Can we also find the corresponding functors $\cdot^{\#}, \cdot_{\#}$ defined before? If R, S are finitely generated reduced k-algebras and $f : R \to S$ is a k-algebra homomorphism, then the pre-image of a maximal ideal is maximal, which defines a map $f_{\#} : Spec_m S \to Spec_m R$ such that $m \mapsto f^{-1}(m)$. This is the same map as before.
A note on schemes: If R is a commutative ring, but not necessarily reduced or finitely generated k-algebra, then we can still define and study $Spec_m(R)$. The problem here is that the pre-image of a maximal ideal under the ring-homomorphism $f : R \to S$ is not necessarily maximal, so we can't define the category. But pre-image of a prime ideal is always prime!! So we define

$$Spec(R)\{p \subset R | p \text{ is prime ideal}\}$$

and the same topology, and we call it a scheme, and study the geometry of that. This is the fundamental idea of Grothendieck from the 1960's. It allows to study arithmetic questions via arithmetic rings (like $\mathbb{Z}$) and their spectra.

Example. What is $Spec(\mathbb{Z})$? What is the topology on it?

6 Primary Decomposition

Let R be a Noetherian ring, typically $R = k[x_1, \ldots, x_n]/J$ is the coordinate ring of a variety. If $I \subseteq R$ is a radical ideal, then by the Nullstellensatz

$$I = P_1 \cap \ldots \cap P_r$$

is the intersection of prime ideals corresponding to the irreducible components of $\mathbb{C}(I)$. What is I is not radical? Do we have a similar "decomposition into smaller pieces" of $\mathbb{V}(I)$?

Definition 6.1. An ideal $Q \subsetneq R$ is primary if for $f, g \in R$

$$fg \in Q \Rightarrow f \in Q \text{ or } g^n \in Q \text{ for some } n > 0.$$

Equivalently, if all zero devisors of $R/Q \neq 0$ are nilpotent.

The radical $P = \mathrm{rad}(Q)$ of a primary ideal is prime:

$$fg \in P \Rightarrow f^n g^n \in Q \text{ for some } n > 0 \Rightarrow f^n \in Q \text{ or } g^{nm} \in Q \Rightarrow f \text{ or } g \in P.$$

Definition 6.2. The primary ideal Q is called P-primary if $P = \mathrm{rad}(Q)$. A *primary decomposition* of an ideal $I \subset R$ is an expression

$$I = Q_1 \cap \ldots \cap Q_k$$

with each Q_i primary. This is the shortest primary decomposition of I if

1. $I \nsubseteq \cap_{i \neq j} Q_i$ for any $1 \leq j \leq k$
2. Q_i is P_i-primary with $P_i \neq P_j$ for $i \neq j$.

Note that if Q_1, Q_2 are P-primary ideal then so is $Q_1 \cap Q_2$, so we can easily get a shortest decomposition from any given one by grouping together the Q_i's with the same radical.

Proposition 6.3. *In a Noetherian ring R every ideal has a primary decomposition*

Proof. Let's call an ideal $I \subset R$ indecomposable if

$$I = J \cap K \text{ with } J, K \text{ ideals } \Rightarrow I = J \text{ or } I = K$$

Note that every prime ideal is indecomposable.
Step 1: In R every ideal is the intersect of indecomposables.
Indeed, let $\mathcal{S}$ denote the set of ideals in R which are not expressible as intersection of

indecomposables. Assume that $\mathcal{S} \neq \emptyset$. Then there is a maximal element of $\mathcal{S}$ (since R is Noetherian), say I, which cannot be indecomposable, but then $I = J \cap K$ with J, K strictly bigger ideals. But then $J, K \notin \mathcal{S}$ and therefore J, K are intersections of indecomposables and therefore so is I.
Step 2: In R every indecomposable ideal is primary.
Note that $Q \subset R$ is indecomposable $\Leftrightarrow$ $0 \subset R/Q$ is indecomposable, and the same for primary ideals, so it is enough to prove that if R is a Noetherian ring then

$0 \subset R$ indecomposable $\Rightarrow 0 \subset R$ primary.

To prove this let $x, y \in R$ with $xy = 0$. Then $y \in \mathrm{Ann}(x)$. Consider the chain

$$\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \ldots \subset \mathrm{Ann}(x^n) \subset \ldots$$

Since R is Noetherian, $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1})$ for some n. Now $(x^n) \cap (y) = 0$; indeed, if $a = cy \in (x^n) \cap (y)$ then $ax = cxy = 0$, and on the other hand, $a = dx^n$, but then $ax = dx^{n+1}$ so $d \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$, that is $a = dx^n = 0$.
Hence if 0 is indecomposable, $xy = 0$ then x^n or $y = 0$ so 0 is primary. $\square$

Example. Take $I = (y^2, xy) \subset k[x, y]$. A primary decomposition is

$$I = (y) \cap (x, y)^2$$

where (y) is already prime and $(x, y)^2$ is primary to (x, y). This is not unique:

$$I = (y) \cap (x, y^2) \text{ or } I = (y) \cap (x + y, y^2)$$

are primary decompositions too.

Theorem 6.4. *The prime ideals $P_i = \mathrm{rad}(\mathrm{Q_i})$ of any primary decomposition*

$$I = Q_1 \cap \ldots \cap Q_s$$

are uniquely determined, these are called the associated primes *to I and the corresponding reduced components $\mathbb{V}(P_i) \subset \mathbb{V}(I)$ are the associated reduced components. Those components which are set theoretically not maximal are called* embedded components.

Proof. $P_1, \ldots, P_s$ are precisely those prime ideals of R which occur as the annihilator of some point in R/I:

$$P_i = \mathrm{Ann}(x_i) = \{r \in R : rx_i \in I\},$$

and therefore they are uniquely determined by I. $\square$

So the primary decomposition is not necessarily unique, but the associated primes are. In the example above these associated primes are (x) and (x, y). The origin $\mathbb{V}(x, y)$ is an embedded point.

7 Algebraic groups and group actions

Definition. An affine algebraic group is an affine variety with group structure, and structure morphisms. That is the multiplication $m : G \times G \to G$ and inverse map $\cdot^{-1} : G \to G$ are morphisms of affine varieties.

Example.
- Finite groups (discrete points on a variety)
- $SL_n(k) = \mathbb{V}(\det -1) \subset \mathbb{A}^{n^2}$ is an algebraic group.
- $k^* = k \setminus \{0\}$ is an affine variety if we think of it as $\mathbb{V}(xy-1) \subseteq \mathbb{A}^2$
- $k \cong \mathbb{A}^1$ with additive structure

Definition. The action of G on X, is a morphism of affine varieties $G \times X \to X$, with the usual properties for a group acting on a set.

Example. k acts on $\mathbb{A}^2$ such that $t(a,b) = (t^{-1}a, tb)$ for all $t \in k^*$. $O_1 = (0,0)$ is an orbit, so are $O_2 = \{(a,0) : a \in k^*\}$ and $O_3 = \{(0,b) : b \in k^*\}$. Finally there is a one-parameter family of orbits $O(s) = \mathbb{V}(xy-s)$. The closures of O_2, O_3 contain O_1. What sort of quotient will work? Can we think of the quotient as an affine variety?

Definition 7.1. (Categorical Quotient) Let G be an affine algebraic group acting on the affine variety X. Then $F : X \to Y$ is a categorical quotient (F is a morphism, Y is affine) if

1. F is constant on the orbits of the group action
2. If $H : X \to Z$ is another morphism constant on the orbits, then H factors uniquely through F. That is there exists an $F' : Y \to Z$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[dr, "F"'] & Z \\ & Y \arrow[u, "F'"'] \end{tikzcd}$$

Definition. The action of G on X induces a G-action on $A(X)$ via $f^g(x) = f(gx)$.

If $F : X \to Y$ is constant on the G-orbits, then $F^\# : A(Y) \to A(X)^G$, the image sits in the G-invariant subring of $A(X)$. Is the image equal to the ring of invariants? It is, if G satisfies the following property: every representation of G is reducible, i.e. it is the direct sum of irreducible representations. These groups are called *reductive* algebraic groups.

Example. $(\mathbb{C}, +)$ is not reductive: The matrix representatation $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ has a invariant line but it is not the direct sum of two 1-dimensional representations. But finite groups, k^* or $SL(n,k)$ are reductive.

Theorem 7.2. *(deep) If the reductive group G acts on X linearly, that is $g(x+y) = g(x) + g(y)$, then $A(X)^G$ is a finitely generated reduced k-algebra.*

Remark. Note that $(k, +)$ does not satisfy the above.

Lemma 7.3. *If X is an affine variety, and $a, b \in X$ and if $f(a) = f(b)$ for all $f \in A(X)$ then $a = b$.*

Proof. Embed $X \subseteq \mathbb{A}^n$, write $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Then if $a \neq b$, we have $a_i \neq b_i$ for some i and so $x_i(a) \neq x_i(b)$ □

Theorem 7.4. *If G is reductive and acts linearly on X, then the map $\alpha_\# : X \to$ $\mathrm{Spec}_m(A(x)^G)$ associated to the embedding $\alpha : A(X)^G \hookrightarrow A(X)$ is a categorical quotient of affine varieties.*

Proof. First we show that $\alpha_\#$ is constant on the orbits. Assume it is not, and $\alpha_\#(x) \neq \alpha_\#(gx)$ for some $x \in X, g \in G$. By Lemma 7 there is an $f \in A(\mathrm{Spec}_m(A(x)^G)) = A(X)^G$ such that $f(\alpha_\#(x)) \neq f(\alpha_\#(gx))$. But then

$$\alpha(f)(x) = (\alpha_\#^\# f)(x) = f(\alpha_\#(x)) \neq f(\alpha_\#(gx)) = (\alpha_\#^\# f)(gx) = \alpha(f)(gx)$$

which is impossible as $\alpha(f) = f \in A(X)^G$ (α is an embedding!), we get a contrdiction.
Second, we prove the universal property of $\alpha_\#$. Assume that $h : X \to Z$ is constant on orbits. We want to find a morphism $\widetilde{h} : \mathrm{Spec}_m(A(x)^G) \to Z$ that makes

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "\alpha_\#"'] & Z \\ & Y \arrow[u, "\widetilde{h}"'] \end{tikzcd}$$

commutative.
If $f \in A(Z)$ then $h^\# f(x) = f(h(x)) = f(h(gx)) = h^\# f(gx)$ for all $g \in G, x \in X$, as h is invariant. Thus, $h^\# f \in A(X)^G$ so there are morphisms

$$A(Z) \to^{h^\#} A(X)^G \hookrightarrow^{\alpha} A(X)$$

which induces the desired morphism of varieties:

$$X \to \mathrm{Spec}_m(A(x)^G) \to Z.$$

□

Example. **1)** k^* acts on $\mathbb{A}^2$ such that $t(a,b) = (t^{-1}a, tb)$ for all $t \in k^*, (a,b) \in \mathbb{A}^2$. The induced k^*-action on $k[x,y]$ is $t \cdot x = tx$ and $t \cdot y = t^{-1}y$ for all $t \in k^*$. Then $k[x,y]^{k^*} = k[xy] \cong k[w]$ (check this!) with the map $k[w] \hookrightarrow k[x,y]$ such that $w \mapsto xy$. This defines the quotient map $\mathbb{A}^2 \to \mathbb{A}^1$ such that $(a,b) \mapsto ab$. The orbits are $\{xy = s \neq 0\} \mapsto s$, $\{(a,0), a \neq 0\} \mapsto 0$, $\{(0,b), b \neq 0) \mapsto 0$ and $\{(0,0)\} \mapsto 0$
We see that the quotient map is not injective on orbits, in other words: the categorical quotient $\mathbb{A}^1$ is not "the set of orbits". Two orbits map to the same point iff the closure of their orbits has a nontrivial intersection.

2) $\mathbb{Z}_2$ acts on $\mathbb{A}^2$ such that $(-1)(a,b) = (-a,-b)$, inducing the action $(-1)x = -x$ and $(-1)y = -y$ on $k[x,y]$. We have that

$$k[x,y]^{k^*} = k[x^2, y^2, xy] \cong k[z_1, z_2, z_3]/(z_1 z_3 - z_2^2)$$

Let $Y = \mathbb{V}(z_1 z_3 - z_2^2) \subseteq \mathbb{A}^3$. We want a map $\mathbb{A}^2 \to Y$. The map $A(Y) = A(X)^G \hookrightarrow A(X)$ is

$$k[z_1, z_2, z_3]/(z_1 z_3 - z_2^2) \hookrightarrow k[x,y]$$

such that $z_1 \mapsto x^2$, $z_2 \mapsto y^2$ and $z_3 \mapsto xy$. The associated quotient map $\mathbb{A}^2 \to Y$ is $(a,b) \mapsto (a^2, ab, b^2)$.

8 Discrete Invariants

In this section we associate certain integers/polynomials to affine/projective varieties: the dimension, the degree and the Hilbert polynomial. The dimension is a true invariant: isomorphic varieties have the same dimension, which we prove later. The degree and Hilbert polynomial are defined for projective varieties and they depend on the embedding into projective space. For affine varieties they are defined as the degree/Hilbert polynomial of the projective closure.

8.1 Dimension

Definition 8.1. (Geometric) The dimension of an affine (projective) variety X is the maximal length n of a chain of irreducible affine (projective) subvarieties

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X.$$

The local dimension $\dim_x X$ at $x \in X$ is maximal length of a chain starting with $X_0 = \{x\}$.

If X is not irreducible then $X_n \neq X$, and X_n is one of the maximal dimensional irreducible components of X, so the dimesnion of X is equal to the dimension of its maximal dimensional components. The local dimension at x is the dimension of the irreducible component containing x.

For example, if $X = \mathbb{V}(xy, xz) \subset \mathbb{A}^3$ is the union of a plane and a line transverse to it, then the local dimension at the points on the line (apart from the intersection point) is 1, whereas it is 2 at the points on the plane.

Definition 8.2. (Algebraic) The dimension of an affine variety X is the maximal length n of a chain of prime ideals in $A(X)$:

$$\{0\} \subseteq I_0 \subsetneq \ldots \subsetneq I_n \subsetneq A(X)$$

The dimension of a projective variety is the maximal length of a chain of homogeneous prime ideals I_j such that $(x_0, \ldots, x_n) \nsubseteq I_j$

The affine and projective Nullstellensatz imply that the geometric and algebraic definitions coincide. Note that $\{0\} = I_0$ in a longest chain if and only if $A(X)$ is reduced, that is, X is irreducible. Otherwise $A(X)/I_0$ is the coordinate ring of the maximal dimensional irreducible component of X.

Lemma 8.3. *If X, Y are irreducible affine (projective) varieties and $X \subsetneq Y$, then* $\dim X < \dim Y$.

Proof. Comes from the definition: we can extend any chain ending in X with Y. $\square$

Theorem 8.4. $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$

We prove this theorem later using alternative definitions of dimension. We have a chain of length n

$$\emptyset \subset \mathbb{A}^0 \subset \ldots \subset \mathbb{A}^{n-1} \subset \mathbb{A}^n,$$

so $\dim \mathbb{A}^n \geq n$. The other direction is more difficult.

Proposition 8.5. *The affine irreducible variety $X \subseteq \mathbb{A}^n$ has dimension $n-1$ if and only if $X = \mathbb{V}(f)$ for some irreducible $f \in k[x_1, \ldots, x_n]$.*

Proof. First assume X is a $n-1$-dimensional irreducible affine variety. $\mathbb{I}(X) \neq (0)$ so $\mathbb{I}(X)$ contains some polynomial $g \neq 0$ say. Write $g = g_1 \cdots g_k$ as the product of irreducible factors. Since X is irreducible, $\mathbb{I}(X)$ is prime, so $g_i \in \mathbb{I}(X)$ for some i; write $f = g_i$. So $(f) \subseteq \mathbb{I}(X)$ and therefore $X \subseteq \mathbb{V}(f) \subsetneq \mathbb{A}^n$. SInce f is irreducible, $\mathbb{V}(f)$ is an irreducible subvariety. From $\dim \mathbb{A}^n = n$ follows that $\dim \mathbb{V}(f) \leq n-1$. But $\mathbb{V}(f)$ and X are both $n-1$ dimensional irreducible varieties, and $X \subseteq \mathbb{V}(f)$, so by the above lemma we have $X = \mathbb{V}(f)$.

Conversely, if $f \in k[x_1, \dots, x_n]$ is an irreducible polynomial then $\mathbb{V}(f) \subseteq \mathbb{A}^n$ is an irreducible subvariety of dimension $n-1$. Indeed, since $\dim \mathbb{A}^n = n$, we only have to prove that there is no prime ideal J between (f) and (0). Assume J is such a prime ideal:

$$(0) \subsetneq J \subsetneq (f),$$

and let $0 \neq g = f^k h \in J$ be a generator in some generating set of J. Here $k \geq 1$ and h is a polynomial such that f does not divide h. Then $h \notin J$ and we have 2 cases: if $f^k \notin J$ then J is not prime ideal as $hf^k \in J$; if $f^k \in J$ then then since J is prime, it is radical so $f \in J$ a contradiction as $J \subsetneq (f)$. $\square$

Note that the same proof works for the projective case (but with homogeneous polynomials).

8.2 Degree

Definition 8.6. The degree of a projective variety $X \subseteq \mathbb{P}^n$ is the maximal possible finite number of intersections of X with a linear subvariety $L \subseteq \mathbb{P}^n$, where $\dim L + \dim X = n$.

Note that this maximum is "almost always" achieved. We say that for a generic linear subvariety L, where $\dim L + \dim X = n$ this maximum is attained. Generic means that the "bad" L's (where the intersection has less points or higher dimensional) form a closed set in the space of all linear subvarieties, that is, the corresponding Grassmannian.

Example. Take $\mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$, we prove the degree is 2. This is obvious if one considers the general form of a line $L = \mathbb{V}(ax + by + cz)$. If $c \neq 0$ these two will intersect away from $H_x = \mathbb{V}(x)$, where we can set $x = 1$ and if $b \neq 0$ then we get $y = \frac{-cz-a}{b}$. Solutions are $z - \left(\frac{-cz-a}{b}\right)^2 = 0$, that is $-c^2z^2 + (b^2 - 2ca)z - a^2 = 0$. A generic line L in this case means that $b, c \neq 0$ and this quadratic equation in z has 2 roots. That is, the discriminant of this quadratic equation is nonzero: $(b^2 - 2ac)^2 - 4a^2c^2 = b^2(b^2 - 4ac) \neq 0$. In other words, the bad L's form a closed subset $\mathbb{V}(bc(b^2 - 4ac))$.

Recall that $X = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$ and that $X \cong \mathbb{P}^1$ (via the Veronese embedding). However, $\mathbb{P}^1$ has degree 1, and so this shows that the degree is not isomorphism invariant, it depends on the embedding.

Definition. Degree of an affine variety $X \subset U_0 \subset \mathbb{P}^n :=$ Degree of its projective closure $\overline{X} \subset \mathbb{P}^n$.

Theorem 8.7. *Let $F \in k[x_0, \dots, x_n]$ be a homogeneous polynomial with no repeated factors of degree d. Then* $\deg \mathbb{V}(F) = d$.

Proof. Let $L \subseteq \mathbb{P}^n$ be an arbitrary line. Then $X \cap L = \mathbb{V}(F|_L) \subseteq L \cong \mathbb{P}^1$. By a linear change of coordinates we can assume $L = \mathbb{V}(x_2, \ldots, x_n)$ and x_0, x_1 are homogeneous coordinates on L, then $F|_L$ is a degree d polynomial in x_0, x_1 (unless if $F|_L = 0$ in which case we can move L a little, this is a genericity condition). By the fundamental theorem of algebra, this has at most d solutions (with multiplicity) and generically it has d solutions. (and this is the other genericity condition: the discriminant is again, nonzero) $\square$

Theorem 8.8. *(Weak form of Bezout's Theorem) Let $X, Y \subseteq \mathbb{P}^n$ be subvarieties of pure dimension such that* $\dim X \cap Y = \dim X + \dim Y - n$*. Then*

$$\deg X \cap Y \leq \deg X \deg Y$$

8.3 The Hilbert Function for Projective Varieties

Recall that if $X \subseteq \mathbb{P}^n$ then $S(X) = A(\hat{X}) = \oplus_{d \geq 0} S(X)_m$ is a graded ring. The next invariant we define encodes the graded pieces of $S(X)$ and therefore it is not invariant under isomorphism but depends on the embedding of the projective variety into a projective space. It is, however, invariant under projective equivalence, that is, linear change of coordinates on the ambient $\mathbb{P}^n$.

Definition 8.9. The Hilbert function on X is defined as $h_X : \mathbb{N} \to \mathbb{N}$, $h_X(m) = \dim S(X)_m$ as a k-vector space.

Recall that $S(X) = k[x_0, \ldots, x_n]/\mathbb{I}(X)$ so

$$\dim S(X)_m = \dim k[x_0, \ldots, x_n] - \dim \mathbb{I}(X)_m = \binom{m+n}{n} - \dim \mathbb{I}(X)_m.$$

Example.

1. $h_{\mathbb{P}^n}(m) = \binom{m+n}{n} = \frac{(m+n)!}{m!n!}$

2. Plane curve $\mathbb{V}(F) \subseteq \mathbb{P}^2$ where F is of degree d. If $m \geq d$ then $\mathbb{I}(X)_m$ is the set of all the homogeneous polynomials of degree m, such that F divides them. This set is in turn in bijection with polynomials of degree $m - d$. Hence for $m \geq d$ we have that

$$h_X(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = dm - \frac{(d-1)(d-2)}{2} + 1 \qquad (*)$$

   Recall from the Algebraic Curves course the degree genus formula where $\frac{(d-1)(d-2)}{2}$ is the genus of the curve. This gives that $h_X(m) = (*) = dm - g + 1$.

Theorem 8.10. *If $X \subseteq \mathbb{P}^n$ a projective variety, h_X its Hilbert function, then*

1. *There exists a* $p_X \in k[x]$ *and* m_0 *such that for all* $m \geq m_0$, $h_X(m) = p_X(m)$. *This* p_X *is called the Hilbert polynomial of* X. *(depends on the embedding!)*

2. *Every coefficient of the Hilbert polynomial is a discrete invariant associated to* X. *More specifically, we have*

   (a) $\deg p_X = \dim X$

   (b) The leading term is $\frac{\deg X}{(\dim X)!} m^{\dim X}$

In fact, the Hilbert polynomial gives "all discrete data"; fix this, and then see how varieties can vary.

Definition 8.11. A flat family of varieties is $\pi : X \to B$, where $X \subseteq \mathbb{P}^n$, a surjective morphism of projective varieties (B can be quasi-projective and π a morphism of quasi-projective varieties, see next sec ion), such that the fibres $\pi^{-1}(b) := X_b$ all have the same Hilbert polynomial.

Example.

- Let $\phi : \mathbb{P}^n \to \mathbb{P}^1$ such that $[x] \mapsto [f_0(x) : f_1(x)]$, where f_0, f_1 are homogeneous polynomials of the same degree such that $af_0 - bf_1 \neq 0$ for all $a, b \in k$ such that $(a, b) \neq (0, 0)$. We then have that $\phi^{-1}[a : b] = \{x \in \mathbb{P}^n : bf_0(x) - af_1(x) = 0\} = \mathbb{V}(bf_0 - af_1)$, which is a hypersurface of degree d. These have the same Hilbert polynomial (exercise).

- The blow-up map $B_0\mathbb{P}^2 \to \mathbb{P}^2$ of $\mathbb{A}^2$ at the point $\{[0 : 0 : 1]\}$ is an example of a non flat family (for details see Chapter 13) This is the map

$$\mathbb{P}^1 \times \mathbb{A}^2 \subset \mathbb{V}(xt - yz) \to \mathbb{A}^2$$

$$([x : y], (t : u)) \mapsto [t : u]$$

  The fibre $\pi^{-1}([t, u]) = \{([x : xt/u], (t, u))\}$ is a point when $u \neq 0$ but the fibre over $(0, 0)$ is $\pi^{-1}(0, 0) = \mathbb{P}^1 \times \{(0, 0)\} \simeq \mathbb{P}^1$. The dimension, that is the dimension of the Hilbert polynomial jumps at the origin.

9 Quasi-Projective Varieties

Goal: Define a category which contains both affine and projective varieties, but also open subsets of these like $\mathrm{GL}(n, k)$, $k^* = \mathbb{A} \setminus \{0\}$ and $\mathbb{A}^2 \setminus \{(0, 0)\}$.

Definition 9.1. A quasi-projective variety $X \subset \mathbb{P}^n$ is a locally closed subset of projective space in the Zariski topology, that is $X = U \cap Z$ is the intersection of a Zariski closed Z and a Zariski open U in $\mathbb{P}^n$.

The above definition includes the following

- Affine varieties are quasi-projective: if $X \subseteq \mathbb{A}^n \simeq U_0 \subset \mathbb{P}^n$, then the projective closure $\overline{X} \subset \mathbb{P}^n$ is a closed subset and $X = \overline{X} \cap U_0$.
- Projective varieties are quasi-projective: for $X \subset \mathbb{P}^n$ we have $X = X \cap \mathbb{P}^n$.
- Things that are neither affine nor projective: open subsets of the above two, for example $\mathbb{A}^2 \setminus \{0,0\} = U_0 \setminus \{0,0\} = U_0 \cap U \cap \mathbb{P}^2$ where $\mathbb{A}^2 = U_0 \subset \mathbb{P}^2$ and $U = \mathbb{P}^2 \setminus \mathbb{V}(x_1, \ldots, x_n)$ are open subsets of $\mathbb{P}^2$.

We define morphisms as locally polynomial maps as usual. We will see an other (equivalent) definition later.

Definition 9.2. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties, then the map $F : X \to Y$ is a morphism if it is locally polynomial, that is, for any $p \in X$ there exists an open subset $U_p \subseteq X$ such that $p \in U$, and there exist homogeneous polynomials $F_0, \ldots, F_m$ of the same degree such that

$$F|_U(p) = [F_0(p) : \ldots : F_m(p)]$$

It is clear that if X, Y are projective, then this definition is the same as before.
If $X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n, Y \subset \mathbb{A}^m = U_0 \subset \mathbb{P}^m$ are affine, and $f : X \to Y, f(x) = (f_1(x), \ldots, f_m(x))$ is a polynomial map (morphism of affine varieties), let $d = \max_i \deg(f_i)$ denote the maximal degree. Then on U_0 f is the same as the homogeneous map

$$[x_0 : x_1 : \ldots x_n] \mapsto [x_0^d : x_0^d f_1(x_1/x_0, \ldots, x_n/x_0) : \ldots : x_0^d f_n(x_1/x_0, \ldots, x_n/x_0)].$$

So f is a morphism of quasi-projective varieties. This means that a morphism of affine varieties in the old sense is a morphism of quasi-projective varieties, and therefore if $X \subset \mathbb{A}^n$ is isomorphic with $Y \subset \mathbb{A}^m$ in the old sense (that is, there is an invertible polynomial bijection between the two) then they are isomorphic as quasi-projective varieties too. The reverse of this does not hold: $\mathbb{V}(xy - 1) \subset \mathbb{A}^2$ and $\mathbb{A}^1 \setminus \{0\}$ are isomorphic as quasi-projective varieties, but there is no polynomial invertible bijection between them as the following example shows:

Example. Let $X = \mathbb{A}^1 \setminus \{0\}$. $X \subset \mathbb{A}^1 = U_1 \subset \mathbb{P}^1$ via the embedding $t \mapsto [t : 1]$. Let $Y = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2 = U_2 \subset \mathbb{P}^2$ via $(x, y) \mapsto [x : y : 1]$. Define $F : X \to Y$ by $[a : b] \mapsto [a^2 : b^2 : ab]$, so on U_1, F is given by $[t : 1] \mapsto [t^2 : 1 : t] = [t : 1/t : 1] \in Y$. Geometrically, this morphism takes a line missing the point 0 to a hyperbola, taking the positive part of the line to the positive part of the hyperbola.
Its inverse $[x : y : z] \mapsto [x : z]$ is well defined on Y, as $Y \subseteq U_2$ and $[t : 1/t : 1] \mapsto [t : 1]$. Hence, $X \cong Y$ as quasi projective varieties. But there is no invertible polynomial map $X \to Y$, so these are not isomorphic affine varieties in our old sense.

Definition 9.3. A quasi projective variety is affine if it is isomorphic as a quasi projective variety to a Zariski closed subset of some affine space.

This is an extension of the category of affine varieties. For projective varieties we don't get an extension: any quasi-projective morphisms $X \to Y$ between the projective varieties X, Y are by definition locally polynomial, which agrees with our first definition of morphism between projective varieties.

9.1 A Basis for the Zariski Topology

Manifolds locally look like Euclidean spaces. Quasi-projective varieties are locally affine spaces in a similar manner as we show below.

Proposition 9.4. *Let X be an affine algebraic variety and $f \in A(X)$, then $U := X \backslash \mathbb{V}(f)$ is an affine variety and $A(U) = A(X)\left[\frac{1}{f}\right]$.*

Here $A(X)\left[\frac{1}{f}\right] = \{\frac{p}{f^n} : p \in A(X), n \in \mathbb{N}\}$ is the ring of Laurent polynomials in f. This is the localisation $A(X)_f$ of $A(X)$ at the multiplicatively closed subset $\{1, f, f^2, \ldots\}$, see the next section.

Proof. We have $f \in A(X) = k[x_1, \ldots, x_n]/\mathbb{I}(X)$. Let $\widetilde{f}$ be a representative for f, such that $\widetilde{f} \in k[x_1, \ldots, x_n]$. Let $I_{\widetilde{f}} = (\mathbb{I}(X), x_{n+1}\widetilde{f} - 1) \subseteq k[x_1, \ldots, x_{n+1}]$, and let $W = \mathbb{V}(I_{\widetilde{f}}) \subseteq \mathbb{A}^{n+1}$. Define a map

$$F : U \to W,\ (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, 1/\widetilde{f}(a_1, \ldots, a_n)$$

and its inverse is the projection

$$F^{-1} : W \to U,\ (a_1, \ldots, a_{n+1}) \mapsto (a_1, \ldots, a_n).$$

Now F is not polynomial, so it is not a morphism of affine varieties. We claim that it is a morphism of quasi-projective varieties. To see this we identify $\mathbb{A}^n$ with the standard affine chart U_{n+1} in $\mathbb{P}^n : X \subseteq \mathbb{A}^n = U_0 \subset \mathbb{P}^n$ via the embedding $(a_1, \ldots, a_n) \mapsto [1 : a_1 : \ldots : a_n]$. Similarly, $W \subset U_0 \subset \mathbb{P}^{n+1}$ by the embedding $(a_1, \ldots, a_{n+1}) \mapsto [1 : a_1 : \ldots : a_{n+1}]$. Now F agrees everywhere on U with the morphism

$$\overline{F} : \mathbb{P}^n \to \mathbb{P}^{n+1},\ [a_0 : \ldots : a_n] \mapsto [a_0\widetilde{f} : a_1\widetilde{f} : \ldots : a_n\widetilde{f} : a_0^{\deg \widetilde{f}+1}]$$

Indeed, on U neither $\widetilde{f}$ nor a_0 is zero, and we can assume $a_0 = 1$, then

$$[a_0\widetilde{f} : a_1\widetilde{f} : \ldots : a_n\widetilde{f} : a_0^{\deg \widetilde{f}+1}] = [\widetilde{f} : a_1\widetilde{f} : \ldots : a_n\widetilde{f}] = [1 : a_1 : \ldots : a_n : 1/\widetilde{f}]$$

The image of $\overline{F}$ is precisely $W \subset U_0 \subset \mathbb{P}^{n+1}$.

The inverse is the projection

$$\overline{F}^{-1} : U_{n+2} \to \mathbb{P}^n,\ [a_0 : \ldots : a_{n+1}] \mapsto [a_0 : a_1 : \ldots : a_n]$$

is well-defined on U_0 where $a_0 \neq 0$, and it identical to f^{-1} on $W \subset U_0 \subset \mathbb{P}^{n+1}$. Finally,

$$A(W) = \frac{k[x_1, \ldots, x_{n+1}]}{(\mathbb{I}(X), x_{n+1}\widetilde{f} - 1)} \cong \frac{A(X)[x_{n+1}]}{(x_{n+1}f - 1)} = A(X)\left[\frac{1}{f}\right]$$

$\square$

Definition. Let X be a topological space. A collection $\mathcal{B} = \{U_i : i \in I\}$ of open subsets is a basis of the topology if any open subset in X is the union of elements in $\mathcal{B}$.

Theorem 9.5. *The affine open subsets of a quasi-projective variety form a basis. (Recall:open=open subset in the Zariski topology, affine=isomorphic as a quasi-projective variety to an affine variety)*

Proof. Embed $X \subseteq \mathbb{P}^n$. First we show that X has a cover of affine open sets. It is enough to show that this is true for each $X_i = X \cap U_i$. But if $X = Z \cap U$ the intersection of a Zariski closed Z and open U in $\mathbb{P}^n$ then $X_i = X \cap U \cap U_i \subset U_i = \mathbb{A}^n$. If $U \cap U_i = \mathbb{A}^n \setminus \mathbb{V}(G_1, \ldots, G_s)$ and $X = \mathbb{V}(F_1, \ldots, F_r$ then

$$X_i = \mathbb{V}(F_1, \ldots, F_r) \setminus \mathbb{V}(G_1, \ldots, G_s)$$

Then

$$X_i \subset \bigcup_{j=1}^{s} (\mathbb{V}(F_1, \ldots, F_r) \setminus \mathbb{V}(G_j)),$$

which are the complements of the hypersurfaces defined by the restriction to G_j to the closed set $\mathbb{V}(F_1, \ldots, F_r)$. This is an affine variety by Theorem 7.4. So every quasi projective variety has a cover of affine open subsets.
Now, let $U \subset X$ be an open subset. Then U is quasi-projective (any open subset of a quasi-projective is quasi-projective), so applying statement we have just proved for U instead of X we get that U is the union of affine opens. $\square$

10 Regular Functions

We have seen that quasi-projective varieties are locally affine, we can think of them as union of affine varieties glued together, like manifolds are glued from Euclidean spaces. We have already seen in the proof of Proposition 9.4 that, locally, quotients of polynomials can define morphisms of quasi-projective varieties.

Definition 10.1. (Regular functions on affine varieties) Let X be an affine variety, and $U \subseteq X$ an open subset. Then $f : U \to k$ is regular at $p \in U$ if there exist $g, h \in A(X)$ with $h(p) \neq 0$ such that $f = g/h$ in a neighbourhood of p, that is $f|_W = g/h|_W$ for an open $p \in W \subset U$. We say that f is regular on U if f is regular on all $p \in U$.

Definition 10.2. (Regular functions on quasi-projective varieties) Let X be a quasi projective variety, and $U \subseteq X$ an open subset. Then $f : U \to k$ is regular at $p \in X$ if there exists an affine open $W \subseteq U$ (that is a Zariski open subset which is affine as quasi-projective variety) such that $p \in W$ and $f|_W$ is regular at p. We say that f is regular on U if it is regular at all $p \in U$.

Remark. An affine open subset of a quasi-projective variety is a Zariski open subset which is affine as a quasi-projective variety. Don't overmistify this! The complement of a hypersurface is open affine by Proposition 9.4, and they form a basis in the Zariski topology by Theorem 9.5, so we can always assume that our affine opens are complements of hypersurfaces.

Definition 10.3.
1. The ring of functions regular on U is denoted by $\mathcal{O}_X(U)$.
2. The ring of germs of regular functions at $p \in X$ is denoted by $\mathcal{O}_{X,p}$, this is defined as

$$\mathcal{O}_{X,p} = \{\text{Pairs } (f, U) : p \in U, f : U \to k \text{ is regular at } p\} / \sim$$

where $(f, U) \sim (f', U')$ if and only if $f|_{U\cap U'} = f'|_{U\cap U'}$.

Example. Let $X = \mathbb{A}^2$, $U = \mathbb{A}^2 \setminus \mathbb{V}(x)$. Take a $f : U \to k$ such that $f(x, y) = y/x$ (slope function). This is regular on U. In fact

$$\mathcal{O}_X(U) = \left\{ \frac{g(x, y)}{x^n} : n \geq 0, g(x, y) \in k[x, y] \right\} = A(X)[\frac{1}{x}]$$

which is $A(U)$ by Proposition 9.4.

Theorem 10.4. *Let X be an affine irreducible variety. Then $\mathcal{O}_X(X) = A(X)$.*

Proof. $A(X) \subseteq \mathcal{O}_X(X)$ by definition. Indeed, for $f \in A(X)$ $f = f/1$ everywhere on X. To prove the opposite direction $g \in \mathcal{O}_X(X)$. Then for all $p \in X$ there exists a U_p such that $p \in U_p \subseteq X$, an open subset, and $h_p, k_p \in A(X)$ such that $k_p(p) \neq 0$ and $g = h_p/k_p$ on U_p. By Theorem 9.5 the affine open sets form a basis for the topology on X, namely, complements of hypersurfaces form a basis. So shrink U_p to be a complement of a hypersurface, that is $p \in U_{F_p} = X \setminus \mathbb{V}(F_p)$.
Now $X = \bigcup_{p\in X}(X \setminus \mathbb{V}(F_p)) = X \setminus (\cap\mathbb{V}(F_p)) = X \setminus \mathbb{V}(\{F_p\})$. Since $A(X)$ is Noetherian, there exist finitely many $F_1, \ldots, F_m$ such that $\mathbb{V}(\{F_p\}) = \mathbb{V}(F_1, \ldots, F_m)$. Thus, $X = \bigcup_{i=1}^m X \setminus \mathbb{V}(F_i)$ and $g|_{U_{F_i}} = h_i/k_i$ on U_{F_i}.

On $U_{F_i} \cap U_{F_j}$ we have $h_i/k_i = h_j/k_j$ and so $k_j h_i = k_i h_j$ on $U_{F_i} \cap U_{F_j}$, which is a dense open set, since X is irreducible. $X = \overline{U_{F_i} \cap U_{F_j}} \subseteq \mathbb{V}(k_j h_i - k_i h_j)$ and so $k_j h_i = k_i h_j$ on X.
But $\mathbb{V}(k_1, \ldots, k_m) = \emptyset$ as $g \in \mathcal{O}_X(X)$ (indeed, for any $p \in X$ there is a k_i such that $k_i(p) \neq 0$), so by the Nullstellensatz, these polynomials generate $A(X)$ (that is, $A(X) = (k_1, \ldots, k_m)$). So $1 = \sum_{i=1}^m l_i k_i$ for some $l_i \in A(X)$. Therefore, $g|_{U_{F_j}} = \sum_{i=1}^m l_i k_i \frac{h_j}{k_j}$ on U_{F_j}. But $k_i h_j = k_j h_i$ on X implies that $g|_{U_{F_j}} = \sum_{i=1}^m l_i k_j \frac{h_i}{k_j} = \sum l_i h_i$ for all j. Thus $g = \sum_{i=1}^m l_i h_i$ on X and $g \in A(X)$. $\square$

Remark. Theorem 10.4 holds also for reducible varieties, but the proof is quite fiddly. The problem is that the set $U_{F_i} \cap U_{F_j}$ may not be dense if X is reducible, so we don't necessarily have that $h_i/k_i = h_j/k_j \Rightarrow k_j h_i = k_i h_j$.

We can redefine morphisms of quasi-projective varieties as follows

Definition 10.5. $F : X \to Y$ is a morphism of quasi-projective varieties if for all $p \in X$ there exists open affine neighborhoods U of p and V of $F(p)$ such that $F(U) \subseteq V$ and $F|_U$ is given by a collection of functions regular on U

By Theorem 10.4 this means that a morphism of quasi-projective varieties is locally polynomial as we expected, and Definition 10.5 and 9.2 are the same. Note that Definition 10.4 does not depend on the embedding of the variety into projective space, this is a local definition.
In the literature morphisms of quasi-projective varieties are also called regular maps, since locally they are given by regular functions.

Definition 10.6. A morphism $F : X \to Y$ of quasi-projective varieties induces the pull-back map $F_p^{\#} : \mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}$ defined as $F_p^{\#}(U, g) = (F^{-1}(U), g \circ F)$.

Knowing $F_p^{\sharp}$ for all $p \in X$ determines F. This is an exercise on problem sheet 6.

Remark. The local nature of regular functions on a quasi-projective variety X can be summarized as follows:

- For any open $U \subset X$ we have a ring $\mathcal{O}_X(U)$.
- If $U_1 \subset U_2$ then the restriction defines a ring homomorphism $\mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1)$.
- If $f_1 \in \mathcal{O}_X(U_1), f_2 \in \mathcal{O}_X(U_2)$ agree on $U_1 \cap U_2$ then they define a regular function $f \in \mathcal{O}_{U_1 \cup U_2}$, whose restriction to U_1, U_2 are f_1, f_2, resp.

A family of rings (or modules) with these properties is called a **sheaf** on X. A pair $(X, \mathcal{O})$ of a space with a sheaf of rings is called a **ringed space**. This particular sheaf $\mathcal{O}_X$ is called the **structure sheaf**.

10.1 Localisation, local rings

Let R be a commutative, Noetherian ring with unit.

Definition 10.7. A subset $S \subseteq R$ is multiplicatively closed if $S \subset R \setminus \{0\}$ and

1. $a, b \in S$ implies that $ab \in S$
2. $1 \in S$

Example. 1. If R is an integral domain, then $R \setminus \{0\}$ is multiplicatively closed.

2. If $p \subsetneq R$ is a prime ideal, then $R \setminus p$ is multiplicatively closed.
3. If $f \in R$ is not nilpotent, then $\{1, f, f^2, \dots\}$ is multiplicatively closed.

Definition 10.8. $RS^{-1} := (R \times S)/\sim$ where $(r, s) \sim (r', s')$ if there exists a $t \in S$ such that $t(rs' - r's) = 0$. Write (r, s) as r/s, and we can think of elements of RS^{-1} as quotients.

Example. Let $R = k[x, y]/(xy)$, $S = \{1, x, x^2, \dots\}$, then $x(y * 1 - 0 * 1) = 0$ so $y/1 = 0/1 = 0$. This is an example of the importance of t in the definition above.

Definition 10.9. Define addition and multiplication in RS^{-1} as follows:

$$\begin{aligned} \frac{r}{s} + \frac{r'}{s'} &= \frac{rs' + r's}{ss'} \\ \frac{r}{s}\frac{r'}{s'} &= \frac{rr'}{ss'} \end{aligned}$$

The following Lemma is a question on Sheet 3:

Lemma 10.10. *RS^{-1} with the addition and multiplication just defined is a commutative ring, with identify $1/1$. The natural map $R \to RS^{-1}$ such that $r \mapsto r/1$ is a ring homomorphism, with kernel $\{r \in R : \exists s \in S, rs = 0\}$.*

Example. 1. Let R be an integral domain. Then $R(R \setminus \{0\})^{-1} = F_R$, the field of fractions.

2. Let $p \subset R$ a prime ideal, and define $R_p = R(R \setminus p)^{-1} = \{r/s : r \in R, s \notin p\}$. This has only one maximal ideal, namely $\{r/s : r \in p, s \notin p\}$. We call rings with exactly one maxinal ideals **local rings**, and R_p is the **localisation** of R at p.
3. For $f \in R$ drfine $R_f := R(\{1, f, f^2, \dots\})^{-1}$. Example: $R = k[x, y]/(xy), f = x$, then $y/1 = 0$ in R_f so $R_f = k[x, x^{-1}]$.

Proposition 10.11. *For X an affine variety and $f \in A(X)$*

1. *If $X_f := X \setminus \mathbb{V}(f)$, then $\mathcal{O}_X(X_f) = A(X)_f$*

2. *If $p \in X$, let $m_p = \{f \in A(X) : f(p) = 0\}$, which is a maximal and therefore prime ideal of $A(X)$. Then $\mathcal{O}_{X,p} = A(X)_{m_p}$.*

Proof.

1. $A(X)_f = A(X)\left[\frac{1}{f}\right] = A(X_f)$ by Proposition 7.4.
2. If $(U, f) \in \mathcal{O}_{X,p}$ then $f|_U = g/h$ for $g, h \in A(X)$ and $h(p) \neq 0$ which means that $h \notin m_p$ so $f \in A(X)_{m_p}$. Conversely, if $f \in A(X)_{m_p}$ then $f = g/h$ with $h(p) \neq 0$; let $U \subseteq X$ be an open set where $h \neq 0$, then $(U, f) \in \mathcal{O}_{X,p}$. These maps are clearly inverses.

□

10.2 Homogeneous localisation

Definition. Let $R = \oplus_{d \geq 0} R_d$ be a graded ring, and $S \subset R$ a multiplicatively closed subset. The degree of an element $r/s \in RS^{-1}$ is $\deg(r) - \deg(s)$. Define

$$RS_0^{-1} = \{r/s \in RS^{-1} : \deg(r) = \deg(s)\},$$

the degree 0 part of the localised ring. This is a subring.

Notation: $(R_p)_0 = R_{(p)}$, $(R_f)_0 = R_{(f)}$.
Recall that for an affine variety X

$$\mathcal{O}_{X,p} = A(X)_{\mathfrak{m}_p} = A(X)\left[A(X) \setminus \mathfrak{m}_p\right]^{-1}$$

where $\mathfrak{m}_p = \{f \in A(X) : f(p) = 0\}$.

Now, if X is a quasi-projective variety, we have two points of view

1. Take $p \in X$, then there exists a $p \in U$ open affine, and $\mathcal{O}_{U,p} = \mathcal{O}_{X,p}$ by the definition of $\mathcal{O}_{X,p}$.
2. Embed $X \subseteq \mathbb{P}^n$, take $\overline{X} \subseteq \mathbb{P}^n$ its projective closure, then we have that $S(\overline{X})$ is a graded ring. If $\deg r = \deg s$, we then ask if r/s is regular at $p \in X$, and the ring of these is the ring of regular functions at p?

Proposition 10.12. *For $X \subseteq \mathbb{P}^n$ a quasi-projective variety, let $\overline{X} \subseteq \mathbb{P}^n$ be its projective closure and $\mathfrak{m}_p = \left\{f \in S(\overline{X}) : f(p) = 0\right\}$. Take $U_0 \subseteq \mathbb{P}^n$ the standard affine chart as usual and assume that $p \in X$ lies on $X_0 := \overline{X} \cap U_0$, and define $\widetilde{\mathfrak{m}_p} = \{f \in A(X_0) : f(p) = 0\}$. Then*

$$S(\overline{X})_{(\mathfrak{m}_p)} = A(X_0)_{\widetilde{\mathfrak{m}}_p} = \mathcal{O}_{X,p}$$

Proof. From what we've seen so far, it is enough to show that $S(\overline{X})_{(\mathfrak{m}_p)} \cong A(X_0)_{\widetilde{\mathfrak{m}}_p}$. Define a map $S(\overline{X})_{(\mathfrak{m}_p)} \to A(X_0)_{\widetilde{\mathfrak{m}}_p}$ such that

$$\frac{f(x_0,\ldots,x_n)}{g(x_0,\ldots,x_n)} \mapsto \frac{f(1,x_1,\ldots,x_n)}{g(1,x_1,\ldots,x_n)}$$

and a map $A(X_0)_{\widetilde{\mathfrak{m}}_p} \to S(\overline{X})_{(\mathfrak{m}_p)}$ such that

$$\frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)} \mapsto \frac{f(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0})x_0^d}{g(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0})x_0^d}$$

where $d = \max(\deg f, \deg g)$. (this is also called homogenisation) Now the numerator and the denominator have the same degree and are both polynomials. These maps are inverses to each other, which gives that $S(\overline{X})_{(\mathfrak{m}_p)} \cong A(X_0)_{\mathfrak{m}_p}$. □

11 Tangent Spaces and Smooth Points

Definition 11.1. For $F \in k[x_1,\ldots,x_n]$ and $p = (p_1,\ldots,p_n) \in \mathbb{A}^n$ we define

$$d_pF = dF|_p(x-p) = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(p)(x_j - p_j) \in k[x_1,\ldots,x_n]_{(1)}$$

which is a linear form. This is the linear approximation to $F(x)$ in a neighbourhood of p.

Definition 11.2. Let $X \subseteq \mathbb{A}^n$ an affine variety, and let $p \in X$. Suppose that $\mathbb{I}(X) = (F_1,\ldots,F_r)$ with $F_i \in k[x_1,\ldots,x_n]$. Then the tangent space to X at p is defined as $T_pX = \mathbb{V}(d_pF_1,\ldots,d_pF_r) \subseteq \mathbb{A}^n$.

Note that the tangent space is a linear subvariety (cut out by linear forms) of $\mathbb{A}^n$, as we have expected. It does have a distinguished point, namely p, so let that be the origin and think of it as a vector space, and identify it by k^n. It can be proved (see the miniproject) that T_pX at a smooth point is the union of tangent lines at p, as we expect.

Example. Let $X = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$ be the cusp. Then $X = \{(t^2,t^3) : t \in k\}$. Then $d_p(y^2 - x^3) = -3x^2|_p(x-p_1) + 2y|_p(y-p_2)$. So if $p = (t^2,t^3)$ then $d_{(t^2,t^3)}(y^2-x^3) = -3t^4(x-t^2) + 2t^3(y-t^3)$. We can see, that away from $t=0$, this corresponds to "what you would have thought". At $t=0$, we have that $d_{(0,0)}(y^2-x^3) = 0$ and $T_0X = \mathbb{V}(0) = \mathbb{A}^2$. So the origin behaves badly, and we say that it is a singular point of X.

Definition 11.3. We say that $p \in X$ is **smooth** if $\dim T_pX = \dim_p X$, where $\dim_p X$ is the local dimension at p of X, that is the dimension of the irreducible component containing X. If $\dim T_pX > \dim_p X$ we say that p is a **singular** point.
(Recall the local dimension at p is defined as the maximal length of chains $\{p\} \subsetneq X_1 \subsetneq \ldots \subsetneq X_m \subset X$ of irreducible varieties starting at p.)

Theorem 11.4. *Let $X \subseteq \mathbb{A}^n$ be an irreducible affine variety of dimension d. Let $\mathbb{I}(X) = (F_1, \ldots, F_r)$. Then $Sing(X)$ is given by the vanishing in X of the $(n-d) \times (n-d)$ minors of the Jacobian $\left(\frac{\partial F_i}{\partial x_j}\right)$, and in particular is a closed subvariety.*

Proof. Let $p \in X$. Then $T_pX = \mathbb{V}(d_pF_1, \ldots, d_pF_r)$ so, since $d_pF = \sum \frac{\partial F_i}{\partial x_j}(p)(x_i - p_i)$, T_pX is the kernel of

$$\phi_p : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial F_1}{\partial x_1}|_p & \cdots & \frac{\partial F_1}{\partial x_n}|_p \\ \vdots & & \vdots \\ \frac{\partial F_r}{\partial x_1}|_p & \cdots & \frac{\partial F_r}{\partial x_n}|_p \end{pmatrix} \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix}.$$

Therefore $p \in \mathrm{Sing}(X) \Leftrightarrow \dim T_pX = \dim\ker\phi_p > d \Leftrightarrow \dim\ker(\mathrm{Jac}(p)) > d \Leftrightarrow (n-d) \times (n-d)$ minors vanish at p. These minors are polynomials, their vanishing set is a closed subvariety. $\square$

Example. (same cusp as before) Let $X = \mathbb{V}(y^2 - x^3)$. We have that $n = 2$, $d = 1$. The Jacobian of this is $\mathrm{Jac}(p) = (-3p_1^2, 2p_2)$ and $(n-d) \times (n-d)$ minors vanish $\Leftrightarrow$ all entries vanish, ie $p = (0, 0)$.

Recall that for $V = k^n$, a finite dimenensional vector space $V^* = \{L : V \to k | L(\lambda v + \mu w) = \lambda L(v) + \mu L(w)\}$ denotes the dual vector space, the vector space of linear forms on V. A linear map $V \to W$ induces a dual map $W^* \to V^*$ by $f^*(L)(v) = L(f(v))$. Choosing an origin of $\mathbb{A}^n$ we have $\mathbb{A}^n \simeq k^n$ and $(\mathbb{A}^n)^* = k[x_1, \ldots, x_n]_1$, the degree-one part of the coordinate ring (using coordinates x_i on $\mathbb{A}^n$.)

Theorem 11.5. *Let X be an affine variety and let $p \in X$ and let $m_p = \{f/g \in \mathcal{O}_{X,p} : f(p) = 0\} \subseteq \mathcal{O}_{X,p}$ be the maximal ideal of regular functions vanishing at p. Then*

$$T_pX \cong \left(\frac{m_p}{m_p^2}\right)^*$$

where * denotes the dual vector space.

Proof. Fix an embedding $X \subseteq \mathbb{A}^n$ such that $p = 0 \in X$. Then $d_0F = \Sigma_{i=1}^n \frac{\partial F}{\partial x_i}(0)x_i$ is a linear form on $\mathbb{A}^n$. But $\mathbb{A}^n = \mathbb{V}(0) = T_0\mathbb{A}^n$ by definition, so we have a map

$$d_0 : k[x_1, \ldots, x_n] \to (T_0\mathbb{A}^n)^*$$

$$F \mapsto d_0 F$$

of k-vector spaces which is linear: $d_0(\lambda F + \mu G) = \lambda d_0(F) + \mu d_0(G)$. Restrict this map to the maximal ideal $M = \{f \in k[x_1, \ldots, x_n] : f(0) = 0\} = (x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$ to get

$$d_0 : M \to (T_0\mathbb{A}^n)^*$$

This map is surjective since $dx_i = x_i$, and $\{x_1, \ldots, x_n\}$ is a basis of $(T_0\mathbb{A}^n)^*$. The the kernel of this map is M^2, indeed,

$$d_0 F = 0 \Leftrightarrow \frac{\partial F}{\partial x_i}(0) = 0 \ \forall i \Leftrightarrow F = 0 \text{ or all monomials in F have order } \geq 2 \Leftrightarrow F \in M^2.$$

Hence $M/M^2 \cong (T_0\mathbb{A}^n)^*$.
Now we want to restrict this argument to X. Recall we fixed $p = 0$. The embedding $j : T_0X \hookrightarrow T_0\mathbb{A}^n$ induces a surjection $j^* : (T_0\mathbb{A}^n)^* \simeq M/M^2 \to (T_0X)^*$, and composed with d_0 we get

$$j^* \circ d_0 : M \to (T_0X)^*$$

The kernel of $j^* \circ d_0$ is $M^2 + \mathbb{I}(X)$ since $f \in Ker(j^* \circ d_0) \Leftrightarrow d_0F|_{T_0X} = 0 \Leftrightarrow d_0F \in \mathbb{I}(T_0X) \Leftrightarrow d_0F = \sum_{i=1}^r a_i d_0 F_i$ where $\mathbb{I}(X) = (F_1, \ldots, F_r)$ and $a_i \in k[x_1, \ldots, x_n] \Leftrightarrow d_0(F - \sum a_i F_i) + \sum_{i=1}^n (d_0 a_i) F_i(0) = d_0(F - \sum a_i F_i) = 0 \Leftrightarrow F - \sum a_i F_i \in M^2 \Leftrightarrow F \in \mathbb{I}(X) + M^2$.
Now let $\overline{M} = \{f \in A(X) : f(p) = 0\} = M/\mathbb{I}(X)$. Note that $\mathbb{I}(X)$ is a sub(vector)space of M as elements of $\mathbb{I}(X)$ vanish on X whereas elements of M vanish at 0, and therefore

$$(T_0X)^* \cong \frac{M}{M^2 + \mathbb{I}(X)} \cong \frac{\overline{M}}{\overline{M}^2}$$

The last step is localisation, we want to prove that $\overline{M}/\overline{M}^2 \cong m_0/m_0^2$. We use the fact that if R is an integral domain then the map $R \hookrightarrow RS^{-1}$ sending r to $r/1$ is an embedding of R into the localised ring RS^{-1}. Apply this with $R = A(X)$ which is integral domain as X is irreducible:

$$A(X) \hookrightarrow \mathcal{O}_{X,0} = A(X)_{m_0} \ f \mapsto f/1$$

which restricts to

$$\overline{M} \hookrightarrow m_0, \overline{M}^2 \hookrightarrow m_0^2$$

This induces an embedding

$$\phi : \overline{M}/\overline{M}^2 \hookrightarrow m_0/m_0^2$$

ϕ is also surjective: if $f/g \in m_0$, let $c := g(0)$, then $\phi(f/c) - f/g = f/c - f/g = f(1/c - 1/g) \in m_0^2$ (since $f \in m_0, 1/c - 1/g \in m_0$) and so $\phi(f/c) = f/g \in m_0/m_0^2$ so ϕ is surjective, and we get $\overline{M}/\overline{M}^2 \simeq m_0/m_0^2$. $\square$

The last theorem implies that the tangent space only depends on an open neighbourhood of p in X, and is independent of embedding, choice of F_i's, etc. So we may define

Definition 11.6. If X is a quasi-projective variety, $p \in X$, then let $m_p = \{f/g \in \mathcal{O}_{X,p} : f(p) = 0\}$ be the maximal ideal of $\mathcal{O}_{X,p}$. Then $T_pX := \left(\frac{m_p}{m_p^2}\right)^*$.

In practice—since we know that the definition is the same on a neighbourhood of $p \in X$—we choose U be an affine neighborhood of p and calculate T_pU using the Jacobian.

We saw that a morphism $F : X \to Y$ of quasi-projective varieties induces $F^\# : \mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}$. Restrict this map to $m_{F(p)}$, then $g(F(p)) = 0$ implies that $F_p^\# g(p) = 0$, so $F_p^\#(m_{F(p)}) \subseteq m_p$. Clearly $F_p^\#(m_{F(p)}^2) \subseteq m_p^2$ too, so we have an induced map

$$\frac{m_{F(p)}}{m_{F(p)}^2} \to \frac{m_p}{m_p^2}$$

Dualising reverses the directions defining

$$\left(\frac{m_p}{m_p^2}\right)^* \to \left(\frac{m_{F(p)}}{m_{F(p)}^2}\right)^*,$$

a map $T_pX \to T_{F(p)}Y$.

Definition 11.7. The differential of F at p, written dF_p is this map $dF_p : T_pX \to T_{F(p)}Y$.

Caution! We defined both d_pF and dF_p, don't mix these up!

12 Rational Maps and Birational Equivalence

Recall, that if R is an integral domain, then $S = R \setminus \{0\}$ is a multiplicatively closed subset, and RS^{-1} is a field, called the field of fractions of R.

Definition 12.1. If X is affine and irreducible variety (hence $A(X)$ is an integral domain) then we call $k(X) = A(X)(A(X) \setminus \{0\})^{-1}$ the **function field** of X. If X is an irreducible quasi-projective variety, let $U \subset X$ be an affine open subset and define function field as $k(X) := k(X \cap U)$. This is independent of U.

As an exercise prove that the function field in the quasi-projective case is independent of the chosen affine open U.

Remark. 1. If $f/g \in k(X)$ and $g(p) \neq 0$ then f/g is regular at p, so $f/g \in \mathcal{O}_{X,p}$. Moreover $f/g \in \mathcal{O}_X(U_g)$ where $U_g = \{p \in X : g(p) \neq 0\}$. So every element of $k(X)$ is regular on some open subset of X.

2. X is irreducible so $A(X)$ is an integral domain, so for all open subset $U \subseteq X$, and for all $p \in X$, we have that $\mathcal{O}_{X,p}, \mathcal{O}_X(U) \subseteq k(X)$. Indeed, by Proposition 8.11 $\mathcal{O}_{X,p} = A(X)_{m_p} \hookrightarrow (A(X)_{m_p})_{A(x)\mu_p} = k(X)$ (localise all the remaining non-zero elements, and recall that $R \to RS^{-1}$ is injective if R is an integral domain.)

Definition 12.2. Let X be an irreducible quasi-projective variety. A **rational map** $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, γ), where $\emptyset \neq U \subseteq X$ is an open subset and $\gamma : U \to Y$ is a morphism of quasi-projective varieties, and the representatives (U, γ) and (V, η) are equivalent if and only if $\gamma|_{U \cap V} = \eta|_{U \cap V}$.

Remark. 1. The rational map is not necessarily defined everywhere on X, just on an open subset.

2. If (U, γ) is a representative of a rational map from X to $Y \subseteq \mathbb{A}^n$, then γ is given by an n-tuple of functions in $\mathcal{O}_X(U)$.

3. Conversely, if $f_1, \ldots, f_n \in k(X)$, then each is regular on a dense open subset. Let U be the intersection of these, then $U \to \mathbb{A}^n$ given by $(f_1, \ldots, f_n)$ is a morphism and so this represents a rational map.

4. Rational maps from reducible varieties: This is the same as a rational map from each irreducible component. So this is not something we work with in general. If $X = \cup X_i$ the union of irreducible components then $k(X) := \oplus k(X_i)$ is a ring but not a field.

In short: rational maps=regular maps defined on some open subset.

Example.
- $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ such that $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_{n-1}]$. Not defined at $[0 : \ldots : 0 : 1]$, so its not a regular map, but it is rational because it is defined on $U_0 \cup \ldots \cup U_{n-1}$.
- If $f = g/h \in k(X)$ then $f : X \dashrightarrow \mathbb{A}^1, x \mapsto f(x)$ is a rational map defined on $U_f = X \setminus \mathbb{V}(h)$.

Composition of rational maps: If $\phi : X \dashrightarrow Y$ and $\eta : Y \dashrightarrow Z$ are rational maps and if there exists a (U, f) representing ϕ and a (V, g) representing η, then if $f^{-1}(V) \neq \emptyset$, we can define the composition as $(U \cap f^{-1}(V), g \circ f)$. So composition of rational maps is not necessarily defined, for example the composition of the projection $\mathbb{A}^2 \to \mathbb{A}^2, (x_1, x_2) \mapsto (x_1, 0)$ and the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{A}^1, (x_1, x_2) \to x_1/x_2$ is not defined.

Definition 12.3. A rational map $F : X \dashrightarrow Y$ is dominant if there exists a (U, γ) representing F such that $\overline{\gamma(U)} = Y$.

Note that composition of dominant rational maps is well-defined dominant rational map.

Definition 12.4. (Birational equivalence) The quasi-projective varieties X and Y are **birationally equivalent** if there is $F : X \dashrightarrow Y$ and $G : Y \dashrightarrow X$ such that $G \circ F|_U = Id_U$ for any open $U \subset X$ where $G \circ F$ is defined, and $F \circ G|_V = Id_V$ for any open $V \subset Y$ where $G \circ F$ is defined.

Example.
- $\mathbb{A}^n$ is birational to $\mathbb{P}^n$. See this by taking the maps $(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_n : 1]$, which is regular everywhere and also $[x_0 : \ldots : x_n] \mapsto (x_0/x_n, \ldots, x_{n-1}/x_n)$, which is not defined if $x_n = 0$, hence we take it to be defined on the open set $U_n = \mathbb{P}^n \setminus \mathbb{V}(x_n)$.
- More generally, if $X \subseteq \mathbb{P}^n$ is quasi-projective and irreducible, then X is birational to $\overline{X}$ and also birational to $X_i = X \cap U_i$ for any i such that $X \not\subseteq \mathbb{V}(x_i)$.

Question: Does the rational map $F : X \dashrightarrow Y$ induce a field homomorphism $f^\sharp : k(Y) \to k(X)$? What is the categorical correspondence between homomorphism of function fields and rational maps of quasi-projective varieties?
Recall from Problem Sheet 3, that $A(Y) \hookrightarrow A(X)$ is injective if and only if $F : X \to Y$ is dominant, that is $\overline{F(X)} = Y$. (Nonzero) field homomorphisms are injective, since the kernel is an ideal so must be 0. This will imply that homomorphisms of function fileds correspond to dominant rational maps.

Theorem 12.5. *Let X, Y be irreducible quasi-projective varieties. Then*

1. *Dominant rational maps $F : X \dashrightarrow Y$ induce k-linear field homomorphisms $F^\# : k(Y) \to k(X)$.*
2. *If $\phi : k(Y) \to k(X)$ is a k-linear field homomorphism, then it induces a unique dominant rational map $\phi_\# : X \dashrightarrow Y$ such that $(\phi_\#)^\# = \phi$.*
3. *If $F : X \dashrightarrow Y$ and $G : Y \dashrightarrow Z$ are dominant then $G \circ F$ is dominant and $(G \circ F)^\# = F^\# \circ G^\#$.*
4. *If $\psi : k(Z) \to k(Y), \phi : k(Y) \to k(X)$ are k-field homomorphisms then $(\phi \circ \psi)_\# = \psi_\# \circ \phi_\#$.*
5. *X and Y are birationally equivalent if and only if $k(X)$ is isomorphic to $k(Y)$.*

We start with a lemma.

Lemma 12.6. *If X, Y affine irreducible varieties then $F : X \dashrightarrow Y$ induces an $F^\# : A(Y) \to k(X)$ and this $F^\#$ is injective if and only if F is dominant.*

Proof. Let $g \in A(Y)$ such that $g : Y \to k$ is a polynomial map on Y, so the composition $g \circ F : X \dashrightarrow k$ is a rational map. We define this to be $F^\# g \in k(X)$. Now let (U, f) be a representative for F. Then $F^\# g$ is defined on U and for $g \in A(Y)$
$F^\# g = 0 \Leftrightarrow g(f(u)) = 0$ for all $u \in U \Leftrightarrow f(u) \in \mathbb{V}(g) \subseteq Y$ for all $u \in U \Leftrightarrow f(U) \subseteq \mathbb{V}(g)$

Hence $F^\#$ injective if and only if $f(U)$ is dense for any U where F is defined. $\square$

Now we prove Theorem 12.5.

Proof. Proof of 1) If X, Y affine, $F^\# : A(Y) \hookrightarrow k(X)$ by the lemma above. Extend this to the field of fractions by defining $F^\#(\frac{f}{g}) = \frac{F^\# f}{F^\# g}$ to get a (nonzero) homomorphism $F^\# : k(Y) \to k(X)$.

If X, Y are quasi-projective, let $(\widetilde{U}, f)$ be a representative for F so $f : \widetilde{U} \to Y$ is a morphism, so there exists a $U \subseteq \widetilde{U}$ affine, and $V \subseteq Y$ affine, and an $f : U \to V$ a morphism of affine varieties which is dominant. So we have $f^\# : k(V) \to k(U)$ defined as above. But $k(V) = k(Y)$ and $k(U) = k(X)$ by definition.

Proof of 2) Suppose first that X, Y are affine and $Y \subseteq \mathbb{A}^m$. Then we have $A(Y) \subset k(Y)$ via the map $f \mapsto f/1$ as usual, so ϕ restricts to an injective homomorphism $\phi : A(Y) \to k(X)$. Let $g_1, \ldots g_r$ be generators of the Noetherian ring $A(Y)$ and $\phi(g_i) = \frac{h_i}{k_i} \in k(X)$ regular on $X \setminus \mathbb{V}(k_i)$. Then for any $f \in A(Y)$ the image $\phi(f)$ is regular on $U = \cap_{i=1}^r (X \setminus \mathbb{V}(k_i)$, that is $\phi(A(Y)) \subset \mathcal{O}_X(U)$. By shrinking to an affine open if needed, we can assume that U is affine open in X, and then by Theorem 10.4 $\mathcal{O}_X(U) = A(U)$ and we have an injective homomorphism $\phi : A(Y) \to A(U)$, which by Theorem 5.12 induces a morphism $\phi_\# : U \to Y$ which is dominant by Lemma 12.6. This is by definition a dominant rational map $\phi_\# : X \dashrightarrow Y$.

When X, Y are quasi-projective then we repeat the same with an affine open $U \subset X$ and affine open $V \subset Y$ using that $k(U) = k(X), k(V) = k(Y)$. We get a dominant rational map $\phi_\# : U \dashrightarrow V$ which represents a dominant rational map $X \dashrightarrow Y$.

Proof of 3) Again, we restrict ourselves to affine X, Y, Z. Let $(U, F|_U)$ represent F and $(V, G|_V)$ represent G, that is, $U \subset X$ open affine, $F|_U : U \to Y$ regular map and $V \subset Y$ open affine, $G|_V : V \to Z$ regular map. Let $\widetilde{U} \subset F^{-1}(V)$ be an open affine. Since the open affines form a basis of the topology, such $\widetilde{U}$ exists. So we have a composition

$$\widetilde{U} \to^{F|_{\hat{U}}} V \to^{G|_V} Z$$

of morphism of affine varieties, and $(\widetilde{U}, G \circ F|_{\widetilde{U}})$ represents the composition $G \circ F$, and $\overline{(G \circ F)(\widetilde{U})} = \overline{G(F(\widetilde{U})} \supseteq \overline{G(\overline{\overline{F(\widetilde{U})}}}) = \overline{G(V)} = Y$ as F and G are dominant, so $G \circ F$ is dominant. Again, using Theorem 5.12 we have $(G|_V \circ F|_{\hat{U}})^\# = F|_{\hat{U}}^\# \circ G|_V^\#$. But for $f/g \in k(Y) = k(V)$

$$F^\#(f/g) = \frac{F|_{\widetilde{U}}^\#(f)}{F|_{\widetilde{U}}^\#(g)}$$

by part 1), and similarly for $G^\#(f/g)$, so for $f, g \in A(Y)$ we have

$$(G|_V \circ F|_{\hat{U}})^\#(f/g) = \frac{(G|_V \circ F|_{\hat{U}})^\#(f)}{(G|_V \circ F|_{\hat{U}})^\#(g)} = \frac{(F|_{\hat{U}}^\# \circ G|_V^\#)(f)}{(F|_{\hat{U}}^\# \circ G|_V^\#)(g)} = (F|_{\hat{U}})^\# \circ G|_V^\#)(f/g).$$

For quasi-projective varieties X, Y, Z choose upen affines and morphisms representing the rational maps F, G and repeat the same argument.
Proof of 4) We saw in 2) that $\phi : k(Y) \to k(X)$ induces a morphism $\phi : A(Y) \to A(U)$ for some $U \subset X$ open, and $\phi_\# : U \to Y$ is the corresponding morphism of affine varieties coming from the equivalence of affine varieties and k-algebras. So there are open affines $U \subset X, V \subset Y, W \subset Z$ such that $\phi_\#, \psi_\#, (\psi \circ \phi)_\#$ are represented by morphisms between these open affines, and $(\phi \circ \psi)_\# = \psi_\# \circ \phi_\#$ follows from Theorem 5.12.
Proof of 5) Again, repeat the argument of Theorem 5.12. Assume X and Y are birational via the rational maps $F : X \dashrightarrow Y, G : Y \dashrightarrow X$. Then $F \circ G|_U = Id|U$ for some open $U \subset X$, so $Id_{k(X)} = Id^\# = (F \circ G)^\# = G^\# \circ F^\#$ is the identity map $k(X) \to k(X)$, so $k(X) \simeq k(Y)$. Conversely, if $\phi : k(Y) \to k(X)$ is an isomorphism then there is an inverse homomorphism $\phi^{-1} : k(X) \to k(Y)$ and $Id_X = Id(k(X))_\# = (\phi \circ \phi^{-1})_\# = \phi_\#^{-1} \circ \phi_\#$, so $\phi_\# : X \to Y$ has an inverse rational map, namely $\phi_\#^{-1}$. $\square$

Definition 12.7. We say that the quasi-projective variety X is **rational** if it is birational to $\mathbb{A}^n$ for some n.

Definition 12.8. If the rational map $F : X \dashrightarrow Y$ is defined on $\emptyset \neq U \subset X$, define the **graph** of F as

$$\Gamma_F = \overline{\{(x, f(x)) : x \in U)\}} \subseteq X \times Y$$

This is independent of the choice of U.

Remark. X is birationally equivalent to Γ_F.

13 Resolution of singularities: Blow-Ups

This is one of the most fundamental ideas in algebraic geometry. Blow ups are widely used in all areas of geometry and topology.

13.1 The Blow-Up of $\mathbb{A}^n$ at the origin

Blowing up a variety at a point means that we replace the point with the projectivisation of the tangent space at that point. We start with blowing up the affine space at the origin. Define

$$\begin{aligned} B_0\mathbb{A}^n &= \left\{(x, l) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x \in l\right\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} \\ &= \left\{((x_1, \dots, x_n), [y_1 : \dots : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x_i y_j - x_j y_i = 0\right\} \end{aligned}$$

This is a closed subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. When $p \in \mathbb{A}^n$ is not the origin we shift the origin to p by the transformation $\rho_p : x \mapsto x - p$ and define

$$B_p\mathbb{A}^n = \mathbb{V}((x_i - p_i)y_j - (x_j - p_j)y_i : 1 \leq i, j \leq n) \subset \mathbb{A}^n \times ¶^{n-1}$$

Define $\pi : B_p\mathbb{A}^n \to \mathbb{A}^n$ by $(x, l) \mapsto x$. This is a **birational morphism**, that is, a morphism which has a rational inverse $\pi^{-1} : \mathbb{A}^n \dashrightarrow B_0\mathbb{A}^n, x \mapsto (x, [x])$ for $x \neq 0$, so π defines an isomorphism between $B_0\mathbb{A}^n \setminus \pi^{-1}(0)$ and $\mathbb{A}^n \setminus \{0\}$.

Definition 13.1. The blowup of $\mathbb{A}^n$ at p is the birational morphism $\pi : B_p\mathbb{A}^n \to \mathbb{A}^n$.

The fibres of $\pi : B_0\mathbb{A}^n \to \mathbb{A}^n$ are

$$\pi^{-1}(x) = \begin{cases} (x, [x]) & \text{if } x \neq 0 \\ \{0\} \times \mathbb{P}^{n-1} & \text{if } x = 0 \end{cases}$$

So π is an isomorphism between $B_0\mathbb{A}^n \setminus \pi^{-1}(0)$ and $\mathbb{A}^n \setminus 0$, but the origin is replaced with $\mathbb{P}^{n-1}$.

Definition 13.2. $E = \pi^{-1}(0) \subset B_0\mathbb{A}^n$ is called the **exceptional divisor**. Points on E are in bijection with lines through p at $\mathbb{A}^n$.

Note that the preimage of the punctured line $L^0 = \{a_1t, \ldots a_nt : t \neq 0\}$ through the origin ($a_i \neq 0$ for some i) is $\pi^{-1}(L^0) = \{(a_1t, \ldots, a_nt), [a_1t : \ldots : a_nt] : t \neq 0\} = \{(a_1t, \ldots, a_nt), [a_1 : \ldots : a_n] : t \neq 0\}$, and therefore $\overline{\pi^{-1}(L^0)} = \{(a_1t, \ldots, a_nt), [a_1 : \ldots : a_n] : t \in k\}$ is a line, corresponding to the point $[a_1 : \ldots : a_n]$ on the exceptional divisor.

13.2 The Blow-Up of $X \subseteq \mathbb{A}^n$ at $p \in X$

Definition 13.3. Define the **strict transform** of X as

$$B_pX = \overline{\pi^{-1}(X \setminus \{p\})} \subseteq B_p\mathbb{A}^n,$$

the closure of $\pi^{-1}(X \setminus \{p\})$ in $B_p\mathbb{A}^n$. The blowup of X at p is the projection map $\pi|_{B_pX} : B_pX \to X$. The exceptional divisor is $E = \pi^{-1}(p) \subset B_pX$.

Note that $\pi : B_pX \to X$ defines an isomorphism between $B_pX \setminus E$ and $X \setminus \{p\}$, and therefore π is a birational morphism. The total preimage in $B_p\mathbb{A}^n$

$$\pi^{-1}(X) = E \cup B_pX \subset B_p\mathbb{A}^n$$

is called the **total transform** of X. It is the union of the exceptional divisor $E = \pi^{-1}(p)$ in $B_p\mathbb{A}^n$ and the strict transform B_pX.

Example. Let $X = \mathbb{V}(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$. Blow up X at 0; to start, clearly we have that $B_0X \setminus \pi^{-1}(0) = \big\{((x,y),[a:b]) : y^2 - x^3 - x^2 = 0, xb = ya, (x,y) \neq (0,0)\big\} \subseteq \mathbb{A}^2 \times \mathbb{P}^2$. To find the points on the exceptional divisor $\pi^{-1}(0)$, let $x_n \to 0$ and $y_n \to 0$ on the curve X. Now for $x_n \neq 0$, we see that

$$\left(\frac{y_n}{x_n}\right)^2 - x_n - 1 = 0 \tag{1}$$

and so, as $x_n \to 0$ we see that $\left(\frac{y_n}{x_n}\right)^2 \to 1$. Thus the extra points in B_0X are $((0,0),[1:1])$ and $((0,0),[1:-1])$.
Now we want to describe B_0X as a quasiprojective variety. $B_0X \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. Assume that $((x,y),[0:1]) \in B_0X$ then $x = 0$; but this implies $y = 0$ and this is a contradiction as $((0,0),[0:1]))$ is not in the exceptional divisor, as seen above. Thus if U_a is the coordinate patch

$$U_a := \{[a:b] : a \neq 0\} \subset \mathbb{P}^1$$

then

$$B_0X \subset \mathbb{A}^2 \times U_a \cong \mathbb{A}^3$$

We make this identification, mapping $((x,y),[a:b])$ to $\left(x, y, \frac{b}{a}\right)$; so $x, y, z = b/a$ are coordinates on $\mathbb{A}^2 \times U_a$.
On $\mathbb{A}^2 \times U_a$ $xb = ya$ transforms to $y = xz$, and we have the original $y^2 - x^3 - x^2$. Substituting this gives $(xz)^2 - x^3 - x^2$, which decomposes, so

$$\pi^{-1}(X) = \mathbb{V}(y - xz, y^2 - x^3 - x^2) = \mathbb{V}(x,y) \cup \mathbb{V}(y - xz, z^2 - x - 1)$$

. The first component is the exceptional divisor, and we get

$$B_0X \cong \mathbb{V}(zx - y, z^2 - x - 1)$$

Note, that via the projection to $\mathbb{A}^2$, B_0X is isomorphic to the quadric plane curve $\mathbb{V}(z^2 - x - 1) \subset \mathbb{A}^2$.

13.3 Blowing up along an ideal

So far we have blown up an affine variety at one point only. We will introduce the notion of blowing up along an ideal gradually in the following steps, summarising what we know about blow-ups at a point first, and then the correspondence between this and blowing up along an ideal.

1. Points $p \in X$ are in bijection with maximal ideals $m_p \subseteq A(X)$, this is the Nullstellensatz.

2. When $p = 0$ the origin, $m_0 = (x_1, \dots, x_n) \cdot A(X) \subset A(X)$ so the x_i's form a generating set of the maximal ideal. We can think of the generators $x_1, \dots, x_n$ as coordinates on the tangent space T_pX.
3. $B_0X = \Gamma_\phi$ is the graph of the rational morphism $\phi : X \dashrightarrow \mathbb{P}^{n-1}$, $x \mapsto [x]$.
4. We have a birational morphism, the projection $B_0X = \Gamma_\phi \to X$. This is an isomorphism between $B_0X \setminus \pi^{-1}(0)$ and $X \setminus \{0\}$.

We will copy this definition for any ideal $I \subset A(X)$.

1. Take any ideal $I \subseteq A(X)$, not necessarily radical ideal!
2. Choose $F_1, \dots, F_r$ generators for I.
3. Define the rational map $F : X \dashrightarrow \mathbb{P}^{r-1}$ such that $x \mapsto [F_1(x) : \dots : F_r(x)]$. This is defined on $X \setminus \mathbb{V}(I)$. Let $B_IX = \Gamma_F$.
4. We have a birational morphism, the projection $\Gamma_\phi \to X$, and π is an isomorphism between $B_IX \setminus \pi^{-1}(X)$ and $X \setminus \mathbb{V}(I)$.

It is important to see (but we don't prove here) that

- B_IX is independent of the choice of generators of I.
- B_IX does depend on the ideal I, not just on $\mathbb{V}(I)$. For example $B_{(x^2,y)}\mathbb{A}^2$ is singular but $B_{(x,y)}\mathbb{A}^2$ is smooth, but the vanishing set of these are the same, the origin.

This leads naturally to the following

Definition 13.4. For any ideal $I \subseteq A(X)$, and set of generators $I = (F_1, \dots, F_r)$ define B_IX to be the graph Γ_F of the rational map $F : X \dashrightarrow \mathbb{P}^{r-1}, x \mapsto [x]$ for $x \notin \mathbb{V}(I)$. The blowup of X at the ideal I is the projection map $B_IX \to X, (x, [x]) \mapsto x$. The exceptional divisor is defiend as $E = \pi^{-1}(\mathbb{V}(I)) \subset B_IX$.
If $Y \subseteq X$ is a closed subvariety, we define B_YX to be $B_{\mathbb{I}(Y)}X$, the blow-up along the radical ideal $\mathbb{I}(Y)$.

The term divisor means a subvariety of codimension one in this context. Indeed, the codimension of E in B_IX is always 1.

13.4 Blowing Up General Quasi-Projective Varieties

Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety. Then $\overline{X} \subseteq \mathbb{P}^n$ is projective, and $S(\overline{X})$ is its homogeneous coordinate ring.

Definition 13.5. If $I \subseteq S(\overline{X})$ a homogeneous ideal, let $F_1, \dots, F_r$ be generators of the same degree.(in the proof of Prop. 4.3 we have seen that there is such a generating set) Define $F : \overline{X} \dashrightarrow \mathbb{P}^{r-1}, x \mapsto [F_1(x) : \dots : F_r(x)]$. Then define $B_I\overline{X} = \Gamma_F \subseteq \overline{X} \times \mathbb{P}^{n-1}$ with map $\pi : \Gamma_F \to \overline{X}$ the natural projection and we define $B_IX = \Gamma_F \cap (X \times \mathbb{P}^{n-1})$

with natural morphism to X. If $\overline{Y} \subseteq \overline{X}$ is a closed subvariety, then $B_{\overline{Y}}\overline{X} = B_{\mathbb{I}(\overline{Y})}\overline{X}$ and $B_Y X = B_{\overline{Y}}\overline{X} \cap (X \times \mathbb{P}^{n-1})$.

13.5 Hironaka's Desingularisation Theorem

In 1964 Hironaka proved a fundamental theorem stating that any quasi-projective variety can be desingularized, i.e, birationally equivalent to a smooth projective variety. His work was recognized with Fields medal in 1970.

Theorem 13.6. *(Hironaka) Let k be a field of characteristic 0. If X is a quasi-projective variety then there exists a smooth quasi-projective variety $\widetilde{X}$, and a birational morphism $\pi : \widetilde{X} \to X$ such that π induces an isomorphism between $\widetilde{X} \setminus \pi^{-1}(X^{smooth}$ and X_{smooth}, and if X is projective, then so is $\widetilde{X}$.*

Hironaka uses blow-ups to construct this smooth variety $\widetilde{X}$. He proves that for an affine variety X there exist an ideal (not necessarily radical ideal!) $I \subset A(X)$, such that $B_I X$ is smooth and $\mathbb{V}(I) = \mathrm{Sing}(X)$, the singular locus of X and π defines the isomorphism between $B_I(X) \setminus \pi^{-1}\mathbb{V}(I)$ and $X \setminus \mathbb{V}(I)$.

13.6 Classification of quasi-projective varieties

In an utopian world we would (will???) be able to describe all varieties up to isomorphism. One ultimate goal of algebraic geometry is to produce a "dictionary of varieties", with a list of all varieties.
However, this is hopelessly difficult, so intead of isomorphism classes first we can try to descripe the birational classes, i.e classify varieties up to birational equivalence. By our Theorem 12 this is the classification of finitely generated fields over k.
Due to Hironaka's theorem there is a smooth representative of each birational equivalence class, but unfortunately this is not unique. Is there a canonical representative (model) of each birational class? This question engines research which motivated great results and has provided a number of Fields medalists in algebraic geometry (Mumford 1974, Deligne 1978, Mori 1990).
As a step towards this classification we prove the following

Theorem 13.7. *Every irreducible variety is birational to a hypersurface (for a field of any characteristic).*

Recall that if X is an irreducible hypersurface then $X = \mathbb{V}(F) \subseteq \mathbb{A}^n$, where F is irreducible. Then $Sing(X) = X \cap \mathbb{V}\left(\frac{\partial F}{\partial x_1}, \dots \frac{\partial F}{\partial x_n}\right)$. This cannot be all of X. (see mini-project). We will use the following result in Galois theory, for the proof see Hulek's book.

Theorem 13.8. *If K is the field of fractions of $k[a_1,\ldots,a_n]$, then the extension K/k can be decomposed as $k \subset K_0 \subset K$, where $K_0 = k(y_0,\ldots,y_m)$ is purely transcendental and $K = K_0[y_{m+1}]/(f)$ is algebraic and separable, where f is irreducible in $K_0[y_{m+1}]$.*

Now we prove Theorem 11.7.
Proof. Every irreducible variety is birationally equivalent to an affine, so we assume X is affine. By the above theorem, $k(X)/k$ decomposes as $k(y_1,\ldots,y_m) = k_0 \subset k_0[y_{m+1}]/(f)$ for some irreducible $f \in k_0[y_{m+1}]$. So $f = y_{m+1}^N + a_{N-1}y_{m+1}^{N-1} + \ldots + a_0$ for some $N \in \mathbb{N}$ and $a_i \in k_0 = k(y_1,\ldots,y_m)$.
We may multiply through the denominators of the a_i (as elements of $k(y_1,\ldots,y_m)$ and get $\widetilde{f} = b_N y_{m+1}^N + b_{N-1}y_{m+1}^{N-1} + \ldots + b_0$, where $b_i \in k[y_1,\ldots,y_m]$. But now $\widetilde{f} \in k[y_1,\ldots,y_{m+1}]$. Let $Y = \mathbb{V}(\widetilde{f}) \subseteq \mathbb{A}^{m+1}$. Then $A(X) = k[y_1,\ldots,y_{m+1}]/(\widetilde{f})$ and $\widetilde{f}$ is irreducible in $k[y_1,\ldots,y_{m+1}]$ since f is irreducible in $k_0[y_{m+1}]$. So Y is an irreducible hypersurface and $k(Y) = k(y_1,\ldots,y_m)[y_{m+1}]/(f) = k(X)$. Finally $k(Y) = k(X)$ and so X and Y are birational. $\square$

Remark. We have that Y is a hypersurface in $\mathbb{A}^{m+1}$ so $\dim Y = m$. Where did m come from though? We started with a field $k(X) = k(Y)$ and we presented it as $k(y_1,\ldots,y_m)[y_{m+1}]/(f)$ so m is the transcendence degree of $k(Y)/k$.

Definition 13.9. (Dimension revisited) The dimension of a variety X is the transcedence degree of $k(X)/k$.

The two definitions we have now are equivalent, but this requires a lot of commutative algebra to prove.