C3.4 ALGEBRAIC GEOMETRY - WEEK 0 SHEET (NOT TO HAND IN)

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Exercise 1. Varieties: solution sets of polynomials.

Let V_0, V_1, V_2 be the solution sets respectively of the three equations

$$y^2 = x^3 \qquad y^2 = x^3 + x \qquad y^2 = x^3 + x^2.$$

Draw in $\mathbb{R}^2$ the solutions, and check that V_0 has a cusp at 0, V_1 has a vertical tangent at 0, and V_2 self-intersects itself at 0. Now work in $\mathbb{C}^2$, what complex solutions are missing?[1]

Exercise 2. Blow-ups.

The *blow-up* of V_2 at $(0,0)$ is defined as the solution set[2]

$$\widetilde{V_2} = \{((x,y),[z_0,z_1]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : y^2 - x^3 - x^2 = 0, xz_1 = yz_0\}.$$

Intuitively, the $\mathbb{CP}^1$ keeps track of the slope $y/x = z_1/z_0$. Show that projection $\widetilde{V_2} \to V_2 \subset \mathbb{C}^2$ to the first factor is a bijection except over $(0,0)$. Does the curve $\widetilde{V_2}$ self-intersect?

Exercise 3. $\mathbb{C}$-algebras.

Let $R = \mathbb{C}[x_1,\ldots,x_n]$ be the ring of polynomials over $\mathbb{C}$ in n variables. Show that a homomorphism $\varphi : R \to S$ of $\mathbb{C}$-algebras[3] is completely determined by the choice of n elements in S, namely the images under φ of $x_1,\ldots,x_n$. Show that S is a finitely generated[4] $\mathbb{C}$-algebra if and only if there is a surjective such $\varphi : R \to S$, for some n. Construct an isomorphism

$$S \cong \mathbb{C}[x_1,\ldots,x_n]/I \quad \text{for some ideal } I \subset \mathbb{C}[x_1,\ldots,x_n].$$

Is this isomorphism unique? (if not, construct a counterexample).

Exercise 4. The functions on a variety.

Consider one of the curves V from Exercise 1, defined by the relevant equation $f = 0$.

Let $\mathrm{Hom}(V,\mathbb{C})$ be the set of all complex functions $V \to \mathbb{C}$ which can be expressed as polynomials over $\mathbb{C}$ in x, y. Check that $\mathrm{Hom}(V,\mathbb{C})$ is a $\mathbb{C}$-algebra.

Consider the $\mathbb{C}$-algebra $\mathbb{C}[V]$, called *coordinate ring of* V, defined by quotienting $\mathbb{C}[x,y]$ by the ideal generated by f,

$$\mathbb{C}[V] = \mathbb{C}[x,y]/(f).$$

Explain why this $\mathbb{C}$-algebra is isomorphic to $\mathrm{Hom}(V,\mathbb{C})$.

The fraction field $\mathbb{C}(V) = \mathrm{Frac}\,\mathbb{C}[V]$ is a field extension of $\mathbb{C}$, and the *dimension* of V is the *transcendence degree* of this extension.[5] Show that our curves V have dimension 1.

Exercise 5. Tangent spaces.

Let V be one of the curves in Exercise 1 defined by the relevant polynomial f. Let $p \in V$. Consider a (complex) line $\ell(t) = p + tv$ through p, parametrized by $t \in \mathbb{C}$, with velocity $v \in \mathbb{C}^2$. The line ℓ is *tangent* to V at p if the polynomial $f(\ell(t))$ in t has a zero of order at least two at $t = 0$. What are the lines tangent to V_0, V_1, V_2?

The *tangent space* T_pV at $p \in V$ is the union of all lines tangent to V at p. Convince yourself that T_pV is a vector space. Say that p is a *singular point* if the vector space dimension $\dim_k T_pV$ of T_pV does not equal $\dim V$ (in our case, $\dim V = 1$). Find the singular points of V_0, V_1, V_2.

Show that by doing a blow-up of V_0 you obtain a curve without singularities.

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[1] *Hint.* how many solutions do you expect if you intersect the curve with $x = c$, some constant?

[2] Recall that the *complex projective line* is $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus (0,0))/\sim$ where we identify $(z_0,z_1) \sim (\lambda z_0, \lambda z_1)$ for any $\lambda \in \mathbb{C} \setminus 0$, and we typically denote the equivalence class by $[z_0 : z_1]$. Notice this space is covered by two open sets: $z_0 \neq 0$ and $z_1 \neq 0$. If $z_0 \neq 0$, we can rescale so that $[z_0 : z_1] = [1 : z]$, so that open set is just a copy of $\mathbb{C}$ parametrized by the variable $z = z_1/z_0$. Similarly $z_1 \neq 0$ is a copy of $\mathbb{C}$ parametrized by $w = z_0/z_1$. The overlap of the two open sets is a copy of $\mathbb{C} \setminus 0$, and there the two parameters are related by $z = 1/w$.

[3] A $\mathbb{C}$-algebra is a ring which is also a vector space over $\mathbb{C}$, satisfying the obvious axioms. A homomorphism of $\mathbb{C}$-algebras is a ring hom (in particular 1 maps to 1) which is also a linear hom of vector spaces over $\mathbb{C}$.

[4] A $\mathbb{C}$-algebra is *finitely generated* by $a_1,\ldots,a_n$ if every element is a polynomial over $\mathbb{C}$ in the $a_1,\ldots,a_n$.

[5] i.e. the maximal number of algebraically independent variables of $\mathbb{C}(V)$ over $\mathbb{C}$. Fact: for an extension $\mathbb{C} \hookrightarrow K$, if $y_1,\ldots,y_m \in K$ are algebraically independent over $\mathbb{C}$, and $\mathbb{C}(y_1,\ldots,y_m) = \mathrm{Frac}\,\mathbb{C}[y_1,\ldots,y_m] \subset K$ is an algebraic extension, then m is the transcendence degree $\mathrm{trdeg}_{\mathbb{C}} K$.