C3.4 ALGEBRAIC GEOMETRY - EXERCISE SHEET 1

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(1) **Zariski topology**
   (a) Verify that arbitrary intersections and finite unions of affine varieties are affine varieties.
   (b) Show that affine algebraic varieties in $\mathbb{A}^n_{\mathbb{C}}$ are closed in the Euclidean topology.
   (c) List the open and closed subsets of $\mathbb{A}^1_{\mathbb{C}}$ (in the Zariski topology).
   Describe briefly[1] the closed subsets of $\mathbb{A}^2_{\mathbb{C}}$.
   (d) Show that the Zariski topology on $\mathbb{A}^2_{\mathbb{C}}$ is *not* the product topology on $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$.

(2) **Irreducibility**
   (a) Show that $\mathbb{A}^n$ is irreducible.
   (b) Show that an affine variety $X \subset \mathbb{A}^n$ is irreducible if and only if every non-empty open subset $U \subset X$ is dense in the Zariski topology.[2]
   (c) Let X be an irreducible affine variety. Show that any two non-empty open sets intersect in a non-empty open dense set.

(3) **Reduced[3] algebras as coordinate rings**
   (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for any ideals I, J.
   (b) Show that the ideal $(xy, xz) \subset k[x, y, z]$ is radical but not prime.
   Draw the variety it defines in $\mathbb{A}^3$.
   (c) Let $X \subset \mathbb{A}^n$ be an affine variety. Show that a radical ideal in $k[X]$ is the intersection of all the maximal ideals containing it.[4]
   (d) Show that a variety $X \subset \mathbb{A}^n$ has two disjoint components if and only if the coordinate ring $k[X]$ may be written as the product of two finitely generated reduced k-algebras.[5]

(4) **The pull-back map between coordinate rings.** Suppose that $F : X \to Y$ is a morphism of affine varieties.
   (a) Show that $F^* : k[Y] \to k[X]$ is injective if and only if F is dominant, i.e. the image set $F(X)$ is dense in Y.
   (b) Show that $F^* : k[Y] \to k[X]$ is surjective if and only if F defines an isomorphism between X and some algebraic subvariety of Y.
   (c) Find an example where F is injective but $F^* : k[Y] \to k[X]$ is not surjective.

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[1] If you wish to do this precisely, there is one tricky part, namely showing that $\mathbb{V}(f, g)$ is a finite set of points when $f, g \in k[x, y]$ have no common factor. *Hints: work over the field $F = k(x) = \operatorname{Frac} k[x]$, and recall that the gcd of f, g over F is an F-linear combination of f, g. Rescaling this to remove denominators, deduce that a k-linear combo of f, g lies in $k[x]$. Can you now bound the possible x-coordinates of points in $\mathbb{V}(f, g)$?*

[2] *Hint: show that X is reducible if and only if there is an open set which is not dense.*

[3] A ring is *reduced* if it has no nilpotent elements except zero. An element r is *nilpotent* if $r^m = 0$ for some $m \geq 1$. Recall an ideal $I \subset R$ is *radical* if $\sqrt{I} = I$, where $\sqrt{I} = \{r \in R : r^m \in I \text{ for some } m \geq 1\}$ is the radical of I. Notice that an ideal $I \subset R$ is radical iff R/I is reduced.

[4] *Hints. Using methods of this course, it is easier to first translate this into a geometrical statement, and prove that. The algebraic proof instead uses a theorem due to Krull: the nilradical $\operatorname{nil}(A) = \{x : x^m = 0$ some $m\}$ of a ring A equals the intersection of all its prime ideals. One applies this to the ring $A = R/I$.*

[5] *Hint. Recall the Chinese Remainder Theorem: if I_1, I_2 are coprime ideals in R (meaning $I_1 + I_2 = R$), then $I_1 \cap I_2 = I_1 \cdot I_2$ and there is a ring isomorphism $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$, $f \mapsto (f + I_1, f + I_2)$.*