C3.4 ALGEBRAIC GEOMETRY - EXERCISE SHEET 2

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(1) **Projective closures and affine cones**

(a) Let X be the parabola $\mathbb{V}(y-x^2) \subset \mathbb{A}^2$. What is its projective closure $\bar{X} \subset \mathbb{P}^2$? Draw the affine cone $\hat{\bar{X}}$ over $\bar{X}$, in $\mathbb{A}^3$, and identify the line corresponding to the "point at infinity" on $\bar{X}$.

(b) Show that the affine varieties $\mathbb{V}(y-x^2) \subset \mathbb{A}^2$ and $\mathbb{V}(y-x^3) \subset \mathbb{A}^2$ are isomorphic. Recalling that $z^2 = x^3$ is a cuspidal cubic with a singularity at zero, can you give an intuitive explanation[1] why those two projective closures in $\mathbb{P}^2$ are not isomorphic?

(2) **The Twisted Cubic.** This is defined to be $C = \mathbb{V}(F_0, F_1, F_2) \subset \mathbb{P}^3$, where

$$\begin{aligned} F_0(z_0,z_1,z_2,z_3) &= z_0z_2 - z_1^2 \\ F_1(z_0,z_1,z_2,z_3) &= z_0z_3 - z_1z_2 \\ F_2(z_0,z_1,z_2,z_3) &= z_1z_3 - z_2^2. \end{aligned}$$

(a) Show that C is equal to the image of the Veronese map,

$$\begin{aligned} \nu : \mathbb{P}^1 &\to \mathbb{P}^3 \\ \nu : [x_0 : x_1] &\mapsto [x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3] \end{aligned}$$

(so ν is given on either coordinate chart by $x \mapsto (x, x^2, x^3)$).

(b) Restrict to the affine patch $U_0 \subset \mathbb{P}^3$ given by setting $z_0 = 1$. Show that $C \cap U_0$ is equal to $\mathbb{V}(f_0, f_1) \subset \mathbb{A}^3$, where $f_i(z_1,z_2,z_3) := F_i(1,z_1,z_2,z_3)$ for $i = 1, 2$.

(c) For $i = 0, 1, 2$ we write Q_i for the quadric surface $\mathbb{V}(F_i) \subset \mathbb{P}^3$. Show that, for $i \neq j$, the surfaces Q_i and Q_j intersect in the union of C and a line L. Therefore no two of them alone may be used to define C.
Deduce that the homogenizations of the generators of an affine ideal do not necessarily generate the homogeneous ideal of the projective closure. (This shows we need to homogenise *all* elements of the affine ideal.)

Cultural Remark: *The codimension of C is 2 in $\mathbb{P}^3$ (it is a curve), but it can be proved that its ideal cannot be generated by 2 polynomials (we have seen that any two of F_0, F_1, F_2 do not generate), so C is not a* complete intersection. *But $C \cap U_i$ is a complete intersection, as we have just seen.*

(3) **Veronese varieties**.

(a) Show that any projective variety is isomorphic to the intersection of a Veronese variety with a linear space.[2]

(b) Deduce that any projective variety is isomorphic to an intersection of quadrics.

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[1] We haven't the tools yet to *prove* these are non-isomorphic, but you should be able to "see" this is true.

[2] Recall a *linear subspace* of $\mathbb{P}^n$ is the projectivisation $\mathbb{P}(V)$ of some k-vector subspace $V \subset k^{n+1}$.
Hint. Use the method with which we studied the image of a projective variety $Y \subset \mathbb{P}^n$ under ν_d.

(4) **Segre embeddings** The image of the Segre morphism $\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \Sigma_{1,1} \subset \mathbb{P}^3$ is known as a "ruled surface".
  (a) What equations define $\Sigma_{1,1}$ as a subvariety of $\mathbb{P}^3$?
  (b) What are the images in $\Sigma_{1,1}$ of $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$? Show that through any point in $\Sigma_{1,1}$ there are two lines lying in $\Sigma_{1,1}$.
  (c) Exhibit some disjoint lines in $\Sigma_{1,1}$. Recall that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma_{1,1}$. Is this isomorphic to $\mathbb{P}^2$? Draw the "real cartoons" of either surface.

(5) **Rational normal curves**
  (a) Let $G(x_0, x_1) = \Pi_{i=1}^{d+1}(b_i x_0 - a_i x_1)$ be a homogeneous degree $d+1$ polynomial with distinct roots $[a_i : b_i] \in \mathbb{P}^1$. Show that $H_i(x_0, x_1) = G(x_0, x_1)/(b_i x_0 - a_i x_1)$ form a basis for the space of homogeneous polynomials of degree d.
  (b) Deduce that the image of the map

$$\mu_d : [x_0 : x_1] \mapsto [H_1(x_0, x_1) : \cdots : H_{d+1}(x_0, x_1)]$$

  is projectively equivalent to the image of the Veronese embedding, that is, it is a rational normal curve.
  (c) What is the image of the point $[a_i : b_i]$? If a_i, b_i are nonzero for all i, what is the image of $[1 : 0]$ and $[0 : 1]$?
  (d) Deduce that through any $d + 3$ points in general position[3] in $\mathbb{P}^d$ there passes a unique rational normal curve.

(6) **Projective variety corresponding to a graded ring.** If $R = \sum_{d \geq 0} R_d$ is a graded ring and $e \geq 1$ is an integer, we define

$$R^{(e)} := \sum_{d \geq 0} R_{de}.$$

We define a grading on $R^{(e)}$ by letting $R_d^{(e)} := R_{de}$.
  (a) Find $k[x_0, x_1]^{(2)}$, expressing it in the form $k[z_0, \ldots, z_n]/I$ for some n and I.
  (b) Find the homogeneous coordinate rings $S(\mathbb{P}^1)$ and[4] $S(\nu_2(\mathbb{P}^1))$. Comment in the context of part (a).
  (c) More generally, show that $S(\nu_e(\mathbb{P}^n)) \cong k[x_0, \ldots, x_n]^{(e)}$, and hence that $k[x_0, \ldots, x_n]^{(e)}$ defines the same projective variety as $k[x_0, \ldots, x_n]$.
  (d) Are $k[x_0, \ldots, x_n]^{(e)}$ and $k[x_0, \ldots, x_n]$ isomorphic as graded k-algebras? Are they isomorphic as (ungraded) k-algebras? What does this imply about the affine cones of $\nu_e(\mathbb{P}^n)$ and $\mathbb{P}^n$?

[3] Meaning, any given $d + 1$ of those points do not lie on a hyperplane in $\mathbb{P}^d$.
[4] Recall that the Veronese morphism $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism onto its image.